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Queueing Networks: Exact Results and Approximations

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Abstract

We prove that in the case of node-independent service rates p, steady-state distributions in discrete-time for a certain class of Jackson networks, can be written as a product of the continuous-time steady state distribution and a factor that approaches one in the limit as service rates go to zero. We conjecture that as $p \to 1$, steady-state distributions in discrete-time are uniformly distributed. We use these results to approximate steady-state distributions for multi-server queueing networks in discrete-time.

1 Introduction

Queueing Networks has become an area of intensive research. Applications of this kind of networks can be found in areas as operation research, computer technology, communications, transportation, electronics, signals processing etc. The theory was developed as an extension of the single station, single server, Poisson stream system. In this system, customers arrive at random at the service station at a mean rate α per unit of time, a Poisson stream, and are served according to a service time that has a negative exponential distribution. The single station case was extended to a system with n nodes, r servers per node, and arrivals and service protocols that respond according to some probability distribution. Jackson [9] was the first to propose a comprehensive theory of queueing networks with his so called product form networks. Jackson developed a general formula for the steady-state distribution for continuos-time networks with Poisson arrivals , exponential service time , and independent routing. Others have developed similar results for different arriving protocols and general arrival and service time probability distributions. Gordon and Newell [8] also produced general results for continuous-time closed and closed cyclic networks. Continuous-time results were used almost exclusively in modeling were queueing theory was applicable. Modeling with Jackson's Product form networks is particularly important because these networks are analogous to a

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system with servers acting independently thus, simplifying the analysis of the system performance.

However, for digital processing devices and computer systems, a discrete-time scale is more appropriate for their modeling. The first explicit results on steady-state behavior of discrete-time closed cyclic networks were published by Pestien and Ramakrishnan in 1994 [[11]. They used what is now called *a direct approach* instead of the traditional time-reversibility approach. Up to date, only two books have been published that formulate comprehensive theoretical results in the single server case: *Queueing Analysis* by Takagi (1996) and *Queueing Networks with Discrete-time scale* by Daduna (2002). Ramakrishnan and Pestien have,among other publications, a series of results for the ample server case. There are not many general results published for the intermediate case in discrete-time , $2 \leq r < k$.

This article develops some exact results and approximations for single, intermediate and ample service closed-cyclic discrete-time networks. The development is guided by Pestien and Ramakrishnan direct approach method. Of particular interest is the approximation of steady-state distributions for the two server case.

This article is dedicated entirely to discrete-time queueing networks. Some new ideas in finding stationary distributions are formulated. The main one shows that the discrete-time stationary distribution can be written in terms of continuous-time distributions times a predictable factor which is proportional to the number of states that were visited in the transition from one state to another. A general result is proven which indicates that in the limit as service probabilities become small, the stationary distribution of the discrete-time network is the same as the analog network in continuous-time. This is an important general result because it provides boundary conditions for the steady state distribution as a function of service probabilities. This result is used here to approximate steady-state distributions in the two server case.

2 Basic Description of Queueing Systems

A queueing system can be described as jobs or customers arriving to a service station to wait for service if this service is not provided immediately After being served, the job leaves the systems. Some example of queuing systems include a customer waiting on line at the bank teller, or a computer program waiting to be run.

Queuing theory provides models to predict the behavior of system performance were the arriving process is random. To develop a model, the characteristics of the system are defined. They include : the arrival mechanism of jobs, the service mechanism of servers, the queue discipline, system capacity, number of server or channels, and the number of service stations or nodes. The arrival process is stochastic, therefore it is necessary to describe it with a probability distribution to predict the time between successively arrivals. There are other characteristic of the arrival process that may have to be described statically that indicate if their is single or bulk arrival and/or if customer are inpatient or wait fro service indefinitely. The time between services is also described with a probability distribution. The description of the service process may need to include the rate at which server work which in turn may depend on the number of jobs present and/or if the server serve a single job or a series of parallel jobs. The queue disciple is generally first come first serve (FCFS) or last come last serve (LCLS).

3 Continuous-time factor: Single Server Case

The invariant distribution for a discrete-time closed-cyclic queueing network with *n-nodes* , *k-jobs* and $r_i = 1$ server per node is given in [**?**] by

$$\pi(s) = G^{-1} \left[\prod_{j \notin OCC(s)} q_j \right] \left[\prod_{j=0}^{n-1} \left(\frac{q_j}{p_j} \right)^{s_j} \right] \tag{1}$$

where

$$G = \sum_{s' \in S} \left[\prod_{j \notin OCC(s')} q_j \right] \left[\prod_{j=0}^{n-1} \left(\frac{q_j}{p_j} \right)^{s'_j} \right]$$

is the normalizing constant and where $OCC(s)$ is the set of indexes $j = 0, \cdots, n-1$ for which node j is not empty. We will show that this distribution can be written in *product form* where one of the factors is the continuous-time invariant distribution for the equivalent network. This will allow as to develop asymptotic behavior results for discrete time queues. The continuous-time distributions for this network is well known [7] and it is given in the following proposition.

Proposition 1 *Consider a continuous-time single-server close cyclic network. Let*

$$\gamma(s_0, s_1, .., s_{n-1}) = \prod_{i=0}^{n-1} \lambda_i^{k-s_i} \tag{2}$$

where $k = \sum_{i=0}^{n-1} s_i$, *and* λ_i *is the service rate at node* i *for* $0 \leq i \leq n-1$. *Then,* γ *is an equilibrium vector for the queue lengths*

Theorem 2 *Consider the following distribution defined on the state space of a discrete-time closed-cyclic network with n-nodes , k-jobs and* $r_i = 1$ *server per node,*

$$\gamma(s) = \left[\prod_{i=0}^{n-1} p_i^{k-s_i} \right] \left[\prod_{i=0}^{n-1} q_i^{s_i - \alpha(s_i)} \right]. \tag{3}$$

where

$$\alpha(s_i) = \begin{cases} 1 \ \textit{if} \ \ s_i > 0 \\ 0 \ \textit{Otherwise} \end{cases}$$

then γ is an invariant vector
Proof. *It suffices to show that γ/π where π is given in (1) is a constant. We have*

$$\frac{\gamma(s)}{\pi(s)}$$

$$= \frac{\left[\prod_{j=0}^{n-1} p_j^{k-s_j}\right]\left[\prod_{j=0}^{n-1} q_i^{s_j-\alpha(s_j)}\right]}{G^{-1}\left[\prod_{j\notin OCC(s)} q_j\right]\left[\prod_{j=0}^{n-1}\left(\frac{q_j}{p_j}\right)^{s_j}\right]}$$

$$= G\frac{\prod_{j=0}^{n-1} p_j^k \cdot \prod_{j=0}^{n-1} p_j^{-s_j} \cdot \prod_{j=0}^{n-1} q_j^{s_j} \cdot \prod_{j=0}^{n-1} q_j^{-\alpha(s_j)}}{\prod_{j=0}^{n-1} q_j^{1-\alpha(s_j)} \cdot \prod_{j=0}^{n-1} q_j^{s_j} \cdot \prod_{j=0}^{n-1} p_j^{-s_j}}$$

$$= G\frac{\left[\prod_{j=0}^{n-1} p_j\right]^k}{\left[\prod_{j=0}^{n-1} q_j\right]^{-1}}, \text{ which is a constant}$$

■

Notice that the first factor in (3) is an invariant vector for the corresponding continuous-time network with transition rate p_i at node i. Thus, the invariant distribution for the discrete-time network may be viewed as the perturbation of the invariant distribution of continuous-time distribution.

We will us this form of the invariant vector to investigate the dynamics in moving from one state to another in a one-step transition.

Example 3 *Consider the state $s = (3242)$ with $k = 11$ and the function $\gamma(s)$ as described in 3. Now consider all possible transitions to s.*

Moves	v	$\gamma(v)$	$\gamma(v) \cdot p(v,s)$	factoring $\gamma(s)$
0	(3242)	$p_0^8 p_1^9 p_2^7 p_3^9 q_0^2 q_1 q_2^3 q_3$	$p_0^8 p_1^9 p_2^7 p_3^9 q_0^3 q_1^2 q_2^4 q_3^2$	$\gamma\left(s\right) q_0 q_1 q_2 q_3$
1	(4142)	$p_0^7 p_1^{10} p_2^7 p_3^9 q_0^3 q_2^3 q_3$	$p_0^8 p_1^{10} p_2^7 p_3^9 q_0^3 q_1 q_2^4 q_3^2$	$\gamma\left(s\right) p_1 q_0 q_2 q_3$
	(3332)	$p_0^8 p_1^8 p_2^8 p_3^9 q_0^2 q_1^2 q_2^2 q_3$	$p_0^8 p_1^9 p_2^8 p_3^9 q_0^3 q_1^2 q_2^3 q_3^2$	$\gamma\left(s\right) p_2 q_0 q_1 q_3$
	(3251)	$p_0^8 p_1^9 p_2^6 p_3^{10} q_0^2 q_1 q_2^4$	$p_0^8 p_1^9 p_2^7 p_3^{10} q_0^3 q_1^2 q_2^4 q_3$	$\gamma\left(s\right) p_3 q_0 q_1 q_2$
	(2243)	$p_0^9 p_1^9 p_2^7 p_3^8 q_0 q_1 q_2^3 q_3^2$	$p_0^9 p_1^9 p_2^7 p_3^9 q_0^2 q_1^2 q_2^4 q_3^2$	$\gamma\left(s\right) p_0 q_1 q_2 q_3$
2	(4232)	$p_0^7 p_1^9 p_2^8 p_3^9 q_0^3 q_1 q_2^2 q_3$	$p_0^8 p_1^{10} p_2^8 p_3^9 q_0^3 q_1 q_2^3 q_3^2$	$\gamma\left(s\right) p_1 p_2 q_0 q_3$
	(4151)	$p_0^7 p_1^{10} p_2^6 p_3^{10} q_0^3 q_2^4$	$p_0^8 p_1^{10} p_2^7 p_3^{10} q_0^3 q_1 q_2^4 q_3$	$\gamma\left(s\right) p_1 p_3 q_0 q_2$
	(3143)	$p_0^8 p_1^{10} p_2^7 p_3^8 q_0^2 q_2^3 q_3^2$	$p_0^9 p_1^{10} p_2^7 p_3^9 q_0^2 q_1 q_2^4 q_3^2$	$\gamma\left(s\right) p_0 p_1 q_2 q_3$
	(3341)	$p_0^8 p_1^8 p_2^7 p_3^{10} q_0^2 q_1^2 q_2^3$	$p_0^8 p_1^9 p_2^8 p_3^{10} q_0^3 q_1^2 q_2^3 q_3$	$\gamma\left(s\right) p_2 p_3 q_0 q_1$
	(2333)	$p_0^9 p_1^8 p_2^8 p_3^8 q_0 q_1^2 q_2^2 q_3^2$	$p_0^9 p_1^9 p_2^8 p_3^9 q_0^2 q_1^2 q_2^3 q_3^2$	$\gamma\left(s\right) p_0 p_2 q_1 q_3$
	(2252)	$p_0^9 p_1^9 p_2^6 p_3^9 q_0 q_1 q_2^4 q_3$	$p_0^9 p_1^9 p_2^7 p_3^{10} q_0^2 q_1^2 q_2^4 q_3$	$\gamma\left(s\right) p_0 p_3 q_1 q_2$
3	(4241)	$p_0^7 p_1^9 p_2^7 p_3^{10} q_0^3 q_1 q_2^3$	$p_0^8 p_1^{10} p_2^8 p_3^{10} q_0^3 q_1 q_2^3 q_3$	$\gamma\left(s\right) p_1 p_2 p_3 q_0$
	(3233)	$p_0^8 p_1^9 p_2^8 p_3^8 q_0^2 q_1 q_2^2 q_3^2$	$p_0^9 p_1^{10} p_2^8 p_3^9 q_0^2 q_1 q_2^3 q_3^2$	$\gamma\left(s\right) p_0 p_1 p_2 q_3$
	(2342)	$p_0^9 p_1^8 p_2^7 p_3^9 q_0 q_1^2 q_2^3 q_3$	$p_0^9 p_1^9 p_2^8 p_3^{10} q_0^2 q_1^2 q_2^3 q_3$	$\gamma\left(s\right) p_0 p_2 p_3 q_1$
	(3152)	$p_0^8 p_1^{10} p_2^6 p_3^9 q_0^2 q_2^4 q_3$	$p_0^9 p_1^{10} p_2^7 p_3^{10} q_0^2 q_1 q_2^4 q_3$	$\gamma\left(s\right) p_0 p_1 p_3 q_2$
4	(3242)	$p_0^8 p_1^9 p_2^7 p_3^9 q_0^2 q_1 q_2^3 q_3$	$p_0^9 p_1^{10} p_2^8 p_3^{10} q_0^2 q_1 q_2^3 q_3$	$\gamma\left(s\right) p_0 p_1 p_2 p_3$

by factoring $\gamma(s)$ we have that for $s_j > 0$,

$$\begin{aligned} & \gamma(v) \cdot p(v,s) \\ = \ & \gamma(s) \cdot \left[\prod_{j:\text{node j received a job}} p_j \right] \cdot \left[\prod_{\text{j:node j did not received a job}} q_j \right] \end{aligned}$$

□

The following lemma , which proves the result in the example above, perhaps provides a better insight into why γ is invariant. As before, let s be the state that results in a one-step transition from state v with movement vector $(m_0, m_1, ..., m_{n-1})$ i.e. for each i , m_i jobs move to node i.

Lemma 4 *Assume a discrete closed cyclic single-server queueing network. Let $\hat{s}$ be a state obtained from $\hat{v}$ in a one step transition. For $s \in S$,*

$$\gamma\left(v\right) P(v,s) = \gamma\left(s\right) \prod_{j=0}^{n-1} p_j^{m_j} \prod_{j=0}^{n-1} q_j^{\alpha(s_j)-m_j}$$

where γ and α are defined by (3).

Proof. $\gamma\left(v\right) \cdot P(v,s)$ equals,

$$\left[\prod_{j=0}^{n-1} p_j^{k-v_j}\right]\left[\prod_{j=0}^{n-1} q_j^{v_j-\alpha(v_j)}\right] \cdot \left[\prod_{j=0}^{n-1} p_j^{m_{j\oplus 1}}\right]\left[\prod_{j:v_j>0}^{n-1} q_j^{1-m_{j\oplus 1}}\right]$$

$$= \prod_{j=0}^{n-1} p_j^{k-(s_j-m_j+m_{j\oplus 1})} \prod_{j=0}^{n-1} q_j^{(s_j-m_j+m_{j\oplus 1})-\alpha(v_j)} \prod_{j=0}^{n-1} p_j^{m_{j\oplus 1}} \prod_{j=0}^{n-1} q_j^{\alpha(v_j)-m_{j\oplus 1}}$$

$$= \left[\prod_{j=0}^{n-1} p_j^{k-s_j}\right]\left[\prod_{j=0}^{n-1} q_j^{s_j-\alpha(s_j)}\right] \cdot \left[\prod_{j=0}^{n-1} p_j^{m_j}\right]\left[\prod_{j=0}^{n-1} q_j^{\alpha(s_j)-m_j}\right]$$

$$= \gamma\left(s\right)\left[\prod_{j=0}^{n-1} p_j^{m_j}\right]\left[\prod_{j=0}^{n-1} q_j^{\alpha(s_j)-m_j}\right]$$

■

Now, let $v \rightarrow s$ indicate that state s resulted for state v in a one-step transition. Let

$$R(v,s) = \left[\prod_{j=0}^{n-1} p_j^{m_j}\right]\left[\prod_{j=0}^{n-1} q_j^{\alpha(s_j)-m_j}\right] \qquad (4)$$

Lemma 5 *Assume a discrete closed cyclic single-server queueing network. Let $R(v,s)$ be defined as (4), then*

$$\sum_{v:v\rightarrow s} R(v,s) = 1$$

Proof. *Consider the expression $\alpha(s_i)$ as defined in (3) and the movement vector $(m_0, m_1, ..., m_{n-1})$ then $m_i \leq \alpha(s_i)$ for all i. So*

$$\sum_{v:v\rightarrow s} R(v,s) = \sum_{m=(m_0,m_1,\ldots,m_{n-1})} \prod_{i=0}^{n-1} p_i^{m_i} \prod_{i=0}^{n-1} q_i^{\alpha(s_i)-m_i} = \prod_{i=0}^{n-1} (p_i+q_i)^{\alpha(s_i)} = 1$$

■

Using Lemma 4, we can provide a very elementary prove showing that γ is invariant.

Theorem 6 *For a single server closed cyclic network, let γ be a vector on S defined by*

$$\gamma\left(s_0, s_1, .., s_{n-1}\right) = \prod_{i=0}^{n-1} p_i^{k-s_i} \prod_{i=0}^{n-1} q_i^{s_i-\alpha(s_i)}$$

where $\alpha(s_i)$ is defined in (2), then γ is an equilibrium vector for P

Proof. $\sum_{v \in S} \gamma(v) p(v,s)$ equals

$$\sum_{v \in S} \gamma(s) \cdot R(v,s) = \gamma(s) \sum_{v:v \to s} R(v,s) = \gamma(s)$$

■

4 Continuous-time factor: Ample Service Case

For the case where there is ample waiting room , $r_i \geq k$ for each i, the discrete-time invariant distribution is actually the same as the continuous-time for an equivalent network. I think this remarkable result was not suggested in the literature because ample service distributions in discrete-time were developed by Ramakrishnan and Pestien fairly recently.

The discete-time invariant distribution is given in [11] by

$$\pi(s) = \frac{k!}{s_0! \cdots s_{n-1}!} \prod_{i=0}^{n-1} \theta_i^{s_i} \tag{5}$$

where

$$\theta_i = \frac{1/p_i}{\sum_{j=0}^{n-1} 1/p_j}$$

Theorem 7 *Consider a closed cyclic network with ample service, then the continuous time invariant distribution is given by the discrete-time invariant distribution (5) with $\lambda_i' s$ replacing $p_i' s$*

Proof. *If π is invariant for Q, then , for the state $(s_0, s_1, \cdots, s_{n-1})$ with $\sum_{i=0}^{n-1} s_i = k$, we must have,*

$$\pi(s_0+1, s_1-1, s_2, \cdots)(s_0+1)\lambda_0 + \cdots + \tag{6}$$

$$\pi(s_0, s_1, \cdots, s_i+1, s_{i\oplus 1}-1, \cdots)(s_i+1)\lambda_i + \cdots +$$

$$\pi(s_0, s_1, \cdots, s_{n-1})(-\lambda_0 s_0 - \lambda_1 s_1 - \cdots - \lambda_{n-1} s_{n-1})$$

$$= \quad 0$$

must be equal to zero. It suffices to show that the discrete-time invariant distribution given in (5) with with $\lambda_i' s$ replacing $p_i' s$ satisfies the expression above *since the invariant distribution is unique. Using (6) notice that*

$$\pi\left(s_0, s_1, \cdots, s_i+1, s_{i\oplus 1}-1, \cdots\right)\left(s_i+1\right)\lambda_i$$

$$= \frac{k!}{s_0!\cdots(s_i+1)!\left(s_{i\oplus 1}-1\right)!\cdots}\theta_0^{s_0}\cdots\theta_i^{s_i+1}\theta_{i\oplus 1}^{s_{i\oplus 1}-1}\cdots\theta_{n-1}^{s_{n-1}}\left(s_i+1\right)\lambda_i$$

$$= \pi\left(s_0, s_1, \cdots, s_{n-1}\right)s_{i\oplus 1}\lambda_i\cdot\frac{\theta_i}{\theta_{i\oplus 1}}$$

Therefore, the left side of (6) becomes

$$\pi\left(s_0, s_1, \cdots, s_{n-1}\right)\left[\sum_{i=0}^{n-1}\lambda_i s_{i\oplus 1}\cdot\frac{\theta_i}{\theta_{i\oplus 1}} - \sum_{i=0}^{n-1}\lambda_i s_i\right]$$

Since (see 5)

$$\frac{\theta_i}{\theta_{i\oplus 1}} = \frac{1/\lambda_i}{1/\lambda_{i\oplus 1}} = \frac{\lambda_{i\oplus 1}}{\lambda_i}$$

Hence, the left-side of (6) further reduces to

$$\pi\left(s_0, s_1, \cdots, s_{n-1}\right)\left[\sum_{i=0}^{n-1}\lambda_{i\oplus 1}s_{i\oplus 1} - \sum_{i=0}^{n-1}\lambda_i s_i\right]$$

which is equal zero. ■

I think this result is important because it says that, with ample waiting room, discrete-time networks can be view as continuous-time networks Therefore, all the performance measures in discrete-time can be analyzed using continuous-time results. In fact , it is now clear why the arrival theorem in discrete-time and continuous-time in the ample service case give us the same results. It is also important to notice that cellular communication is based on the transmission of cells of information of equal length, therefore their analysis is better understood using a discrete-time scale This result may provide some insight in the case where there is ample service in this space.

5 Continuity Results in the intermediate case

The intermediate case denotes the case where $2 \leq r < k$. In this case we think that the continuous-time distribution for the equivalent network is not a multiplication factor of the discrete distribution. The following example compares the two distributions for the case where $n = 2, r_i = 2$

For a closed cyclic network with $n = 2, r_i = 2$, an invariant vector for the Q matrix Q in continuous-time is given by

$$\left\{p_1^3, 2p_1^2p_0, 2p_1p_0^2, p_0^3\right\} \tag{7}$$

In discrete-time an invariant vector for the transition matrix P is given by

$$\{-4p_1^3+4p_1^4-p_1^5+7p_1^3p_o-6p_1^4p_o+p_1^5p_o-4p_1^3p_o^2+p_1^3p_o^3+2p_1^4p_o^2, \tag{8}$$
$$-8p_1^2p_o+8p_1^3p_o-2p_1^4p_o+10p_1^2p_o^2-3p_1^2p_o^3-10p_1^3p_o^2+2p_1^3p_o^3+2p_1^4p_o^2,$$
$$-8p_1p_o^2+8p_1p_o^3-2p_1p_o^4+10p_1^2p_o^2-10p_1^2p_o^3-3p_1^3p_o^2+2p_1^2p_o^4+2p_1^3p_o^3,$$
$$-4p_o^3+4p_o^4-p_o^5+7p_1p_o^3-6p_1p_o^4+p_1p_o^5-4p_1^2p_o^3+2p_1^2p_o^4+p_1^3p_o^3\}$$

Now , if the service rate remains constant from node to node, (7) becomes

$$\{p^3, 2p^3, 2p^3, p^3\}$$

and (8) becomes

$$\{p-1, p-2, p-2, p-1\}$$

After normalizing (7) the unique invariant distribution for Q is

$$\left\{\frac{1}{6}, \frac{2}{6}, \frac{2}{6}, \frac{1}{6}\right\} \tag{9}$$

and after normalizing (8) , the unique invariant distribution for P is

$$\left\{\frac{p-1}{4p-6}, \frac{p-2}{4p-6}, \frac{p-2}{4p-6}, \frac{p-1}{4p-6}\right\} \tag{10}$$

Comparing (9) and (10) we see that it is not possible to factor the continuous-time distribution to get a product form for the discrete-vector. However , in the limit as the rate goes to zero, the two distributions are the same. We will explore this observation farther in the next section.

6 Intermediate Case: Limit Results

The main result of this section is to show that in the limit, the discrete-time invariant distribution approaches the continuos-time counterpart even in the intermediate case ($2 \leq r_i < k$). We will establish one important result that shows that there is a limiting relationship between the $P-matrix$ for the discrete network and $Q-matrix$ for the continuous network. This in turn helps to prove the result mentioned above.

Theorem 8 *Let Q be the transition-rate matrix of a continuous-time cyclic network with k jobs , n nodes , r_i independent servers per node at each node , each with service rate λ. Let P be the transition matrix of a discrete-time cyclic network with k jobs , n nodes , r_i independent servers per node at each node , each with service probability p. Then,*

$$\lim_{p\to 0}\frac{1}{p}[P-I] = \frac{1}{\lambda}Q \tag{11}$$

Proof. Let $o(p)$ denote a function of p satisfying $\lim_{p \to 0} \frac{o(p)}{p} = 0$

Let

$$\alpha_i(s_i) = \begin{cases} s_i & if \quad s_i \le r_i \\ r_i & if \quad s_i > r_i \end{cases}$$

First note that for $q = 1 - p$, we have

$$q^n - 1 = (q-1)\sum_{i=0}^{n-1} q^i = -p\sum_{i=0}^{n-1} q^i$$

Hence

$$\lim_{p \to 0} \frac{1}{p}\left(q^n - 1\right) = -n \tag{12}$$

Now consider any state $s = (s_0, s_1 ..., s_{n-1})$. The system remains in this state after a one-step transition with a probability $q^{\alpha(s_0)+\cdots+\alpha(s_{n-1})} + o(p)$. Hence the s, s^{th} entry of the matrix $(P - I)$ is given by

$$(P-I)_{s,s} = q^{\sum_{i=0}^{n-1}\alpha(s_i)} + o(p) - 1$$

Therefore,

$$\lim_{p \to 0} \frac{1}{p}(P-I)_{s,s}$$

$$= \lim_{p \to 0} \frac{1}{p}\left(q^{\sum_{i=1}^{n}\alpha(s_i)} + o(p) - 1\right)$$

$$= -\sum_{i=0}^{n-1}\alpha(s_i)\ , \qquad \text{using (12) and the definition of } o(p)$$

Hence,

$$= \frac{1}{\lambda} Q_{s,s} \qquad \text{by the definition of } Q \tag{13}$$

Now note that for any state $s' \neq s$ that results in a one-step transition from s only if more than one movement occurs, we have

$$Q_{s,s'} = 0$$

Also,

$$\lim_{p \to 0} \frac{1}{p}(P-I)_{s,s} = \lim_{p \to 0} \frac{1}{p} P_{s,s} = \lim_{p \to 0} \frac{1}{p} o(p) = 0$$

Hence

$$\lim_{p \to 0} \frac{1}{p}(P-I)_{s,s'} = \frac{1}{\lambda} Q_{s,s'} \tag{14}$$

Finally consider a state $s'' \neq s$ which can result from s in a one-step transition with only one movement occurring at node i (where $i \in \{0, 1, \cdots, n-1\}$. Then we have,

$$Q_{s,s''} = \alpha(s_i)\lambda$$

Also,

$$\lim_{p\to 0} \frac{1}{p}(P-I)_{s,s''} = \lim_{p\to 0} \frac{1}{p} P_{s,s''}$$

$$= \lim_{p\to 0} \frac{1}{p}\left[\alpha\left(s_i\right)p + o(p)\right] = \alpha(s_i)$$

Hence,

$$\lim_{p\to 0} \frac{1}{p}(P-I)_{s,s''} = \frac{1}{\lambda} Q_{s,s''} \tag{15}$$

Therefore by (13), (14) and (15) the proof is completed. ■

Theorem 9 *Let π be steady-state distribution in continuos-time for a closed cyclic network with n nodes, k jobs, and r independent servers at each node with service rate λ. Let π_d be the steady-state distribution for the discrete-time network with n nodes, k jobs, and r independent servers per node with service probability p. Then,*

$$\lim_{p\to 0} \pi_d = \pi \tag{16}$$

Proof. As before, let Q be the transition rate matrix for the continuous-time network, and let P be a transition matrix for the discrete-time network , repetitively. Also, we let $\pi_d(p)$ indicate the discrete-time distribution evaluated at p. We will show that, given any sequence $p_n \to 0$, every convergent subsequence of $\{\pi_d(p)\}$ converges to π. This shows that $\lim_{p\to 0} \pi_d$ exist and equals π. We have,

$$\pi_d(p) \cdot P = \pi_d \tag{17}$$

so

$$\pi_d(p)\,(P-I) = 0$$

equivalently,

$$\pi_d(p) \frac{1}{p}\,(P-I) = 0 \tag{18}$$

So, let p_n be a sequence satisfying $p_n \to 0$. If $\{p_{n_i}\}$ is a subsequence such that $\{\pi_d(p_{n_i})\}$ has a limit, say $\hat{\pi}$ then, $\hat{\pi}$ must be a probability vector, since, for each p, $\pi_d(p)$ is also. By theorem 8 we must have

$$\hat{\pi} \frac{1}{\lambda} Q = 0$$

i.e.

$$\hat{\pi} Q = 0 \tag{19}$$

Since π is the unique probability vector that satisfies $\pi Q = 0$, we must have, $\hat{\pi} = \pi$. This completes the proof. ■

7 Asymptotic Behavior as $P \to 1$

We have shown above that for all discrete-time cyclic queueing networks, the steady-state distribution approaches the continuos-time invariant distribution as $p \to 0$. We would like to know the limiting behavior of the discrete distributions as $p \to 1$. Using these two results and the fact that these functions are monotonic, we can develop algorithms for their approximations. This is significant because there are not general formulas for discrete-time invariant distributions in the intermediate case , $2 \leq r < k$. Even in the case of two servers per node, and constant service rates, the distribution are very complex.

Conjecture 10 *As $p \to 1$, in a one step transition, multiple services will be completed with certainty; therefore, if there are sufficiently many jobs , the servers will be occupied at all times since jobs will move to the servers as soon as they are idle. If the number of jobs k is greater than the number of nodes n times the number of servers per node r, $(k > n \cdot r)$, in steady-state, each node i will have r jobs at the r servers and the rest of the jobs will be uniformly distributed.*

An argument for the proof of the conjecture would be as follows: consider a cyclic queueing network with n nodes, k jobs , and r independent servers per node with constant service probability p. Let

$$\alpha(s_i) = \begin{cases} s_i & if \quad s_i \leq r \\ r & if \quad s_i > r \end{cases}$$

Let $s = (s_0, s_1, .., s_{n-1})$ be a state that results from state s' with a movement vector $m_{s'} = (m_0, m_1, ..., m_{n-1})$. then,

$$P(s', s) = \prod_{i=0}^{n-1} p^{m_i} \cdot \prod_{i=0}^{n-1} q^{\alpha(s_i)-m_i},$$

hence, for $0 \leq i \leq n-1$, and $q = 1 - p$

$$\lim_{p \to 1} P(s', s) = \begin{cases} 0 & if \quad m_i < \alpha(s_i) \text{ for any } i \\ 1 & if \quad m_i = \alpha(s_i) \text{ for all } i \end{cases}$$

Now, since π is invariant

$$\sum_{s' \in S} \pi(s') P(s', s) = \pi(s)$$

We can show that for all states s, the limit as $p \to 1$ exist using similar arguments as in theorem 9. Therefore,

$$\lim_{p \to 1} \sum_{s' \in S} \pi(s') \cdot P(s', s) = \lim_{p \to 1} \pi(s)$$

$$\sum_{s' \in S} \lim_{p \to 1} \pi(s') \cdot \lim_{p \to 1} P(s', s) = \lim_{p \to 1} \pi(s)$$

The expression on the left is zero except for those s' that move to s with movement vector $m_{s'} = \left(\alpha(s_0^{'}), \alpha(s_1^{'}), ..., \alpha(s_{n-1}^{'})\right)$. Then, for these s', we have,

$$\sum_{s'} \lim_{p\to 1} \pi(s') = \lim_{p\to 1} \pi(s)$$

If s results form s with a movement vector $\alpha((s_0), \alpha(s_1), ..., \alpha(s_{n-1}))$, the expression above reduces to,

$$\sum_{s'\neq s} \lim_{p\to 1} \pi(s') = 0$$

What this means is that for sufficiently large k, $(k \geq n \cdot r)$, every state s' that reaches s with movement vector $\left(\alpha(s_0^{'}), \alpha(s_1^{'}), ..., \alpha(s_{n-1}^{'})\right)$ is transient and

$$\lim_{p\to 1} \pi(s') = 0$$

Also, s is a recurrent state. So, the proof reduces to showing that the steady-state distribution in the limit as $p \to 1$ is uniform among recurrent states.

In the case where $r = 2$, uniformity of the steady-state as $p \to 1$ could be argued as follows: fix a state $s = (s_0, s_1)$ with $\alpha(s_i) > r$ for $i = 0, 1$. Consider states $v = (s_0 + 1, s_1 - 1)$, then $\alpha(v_i) \geq r$ for $i = 0, 1$. Then, the balance equation is given by,

$$\begin{aligned} &\pi_p(s_0, s_1) \\ = \quad &\pi(s_0, s_1)p^{2r} + \pi(s_0 + 1, s_0 - 1)rp^{2r-1}q + rp^{2r-1}\pi(s_0 + 1, s_0 - 1) + o(q) \end{aligned}$$

Now, due to the symmetry $\pi(s_0 + 1, s_0 - 1) = \pi(s_0 - 1, s_0 + 1)$. So,

$$(1 - p^{2r})\pi_p(s_0, s_1) = 2rp^{2r-1}q\pi(s_0 + 1, s_0 - 1) + o(q)$$

hence

$$\pi_p(s_0, s_1) = \frac{2rp^{2r-1}\pi(s_0 + 1, s_0 - 1)}{(1 + p + p^2 + \cdots + p^{2r-1})} + \frac{o(q)}{q} \cdot \frac{1}{(1 + p + p^2 + \cdots + p^{2r-1})}$$

so, as $p \to 1$,

$$\pi_p(s_0, s_1) = \pi_p(s_0 + 1, s_1 - 1) = \pi_p(v)$$

The above example lead us to believe that the invariant distribution in the limit as $p \to 1$ is uniform.

Example 11 *Assume a network of two nodes and two servers per node. Assume that there are $k = 4$ jobs. The only state with $\alpha(s_i) \geq r$ for $i = 1, 2$ is $(2, 2)$. Therefore, based on the conjecture* $\lim_{p\to 1} \pi(2, 2) = 1$, *and* $\lim_{p\to 1} \pi(s_1, s_2) = 0$ *otherwise.*

The steady-state distribution of this network was calculated directly using the theory of Markov chains (positive recurrent, irreducible). The steady-state distribution as a function of p, is

$$\pi(4,0) = \frac{-7p + 10p^2 - 7p^3 + 2p^4 + 2}{-47p + 60p^2 - 36p^3 + 8p^4 + 16}$$

$$\pi(3,1) = \frac{-12p + 16p^2 - 10p^3 + 2p^4 + 4}{-47p + 60p^2 - 36p^3 + 8p^4 + 16}$$

$$\pi(2,2) = \frac{-9p + 8p^2 - 2p^3 + 4}{-47p + 60p^2 - 36p^3 + 8p^4 + 16}$$

$$\pi(1,3) = \frac{-12p + 16p^2 - 10p^3 + 2p^4 + 4}{-47p + 60p^2 - 36p^3 + 8p^4 + 16}$$

$$\pi(0,4) = \frac{-7p + 10p^2 - 7p^3 + 2p^4 + 2}{-47p + 60p^2 - 36p^3 + 8p^4 + 16}$$

We observe that

$$\lim_{p\to 1} \pi(4,0) = \lim_{p\to 1} \pi(0,4) = 0$$

$$\lim_{p\to 1} \pi(3,1) = \lim_{p\to 1} \pi(1,3) = 0$$

$$\lim_{p\to 1} \pi(2,2) = 1$$

In general for $n = 2, r = 2,$ and k *jobs*, the only states that would not have two jobs in every node are $(k, 0)$, $(0, k)$,$(k-1, 1)$ and $(k, k-1)$. Since there are $k + 1$ states in the state space, based on the conjecture, the steady-state distribution as $p \to 1$ will be uniform for states with $s_0 \geq 2$ and $s_1 \geq 2$. Hence,

$$\lim_{p\to 1} \pi(s_0, s_1) = \frac{1}{k-3}$$

for states with $s_0, s_1 \geq 2$.

8 Approximation of the Queue Length: M/2/2 network with constant service rate

In this section we will study in detail the two nodes, two server per node discrete-time cyclic network. These networks do not have a product form invariant distribution even when the service rate is constant

The distribution for the $M/2/2$ network in continuous-time with constant service rate is given in [8] as

$$\pi(k,0) = \pi(0,k) = \frac{c}{2^{k-1}}$$

and

$$\pi(k-1,k) = \pi(k,k-1) = \frac{c}{2^{k-2}}$$

if $j \geq 2$

$$\pi(k-j,j) = \frac{c}{2^{k-h}2^{h-1}} = \frac{c}{2^{k-2}}$$

Since the sum of the probabilities is 1, then

$$2 \cdot \frac{c}{2^{k-1}} \ + 2 \cdot \frac{c}{2^{k-2}} + (k+1-4)\,\frac{c}{2^{k-2}} = 1$$

so

$$3c + (k-3)\,c = 2^{k-2}$$

or

$$c = \frac{2^{k-2}}{2}$$

Therefore ,

$$\pi(k,0) = \pi(0,k) = \frac{2^{k-2}}{2}\ \frac{1}{2^{k-1}} = \frac{1}{2^k}$$

and

$$\pi(k-1,k) = \pi(k,k-1) = \frac{2^{k-2}}{2}\,\frac{1}{2^{k-2}} = \frac{1}{k}$$

and for $j \geq 2$

$$\pi(k-j,j) = \pi(j,k-j) = \frac{2^{k-2}}{2} \cdot \frac{1}{2^{k-2}} = \frac{1}{k}$$

Using (16) we conclude that for discrete-time networks with two nodes and two servers per node,

$$\lim_{p\to 0} \pi_d(k,0) = \lim_{p\to 0} \pi_d(0,k) = \frac{1}{2k} \tag{20}$$

and for $s_1 \neq 0, s_2 \neq 0$

$$\lim_{p\to 0} \pi(s_1,s_2) = \frac{1}{k}. \tag{21}$$

The behavior of the steady-state distribution as $p \to 1$ can be analyzed using the conjecture above. For large k, since at each one-step transition multiple services take place, the two nodes will be occupied with two jobs at each of their two servers; the rest of the $k-4$ jobs will be uniformly distributed among the nodes. There are $k+1$ states in the state space of this system. Therefore, as the service rate goes to one, the invariant distribution for states with at most one job at a node goes to zero. Hence,

$$\lim_{p\to 1}\pi(k,0)=\lim_{p\to 1}\pi(0,K)=0$$

and

$$\lim_{p\to 1}\pi(k-1,1)=\lim_{p\to 1}\pi(1,K-1)=0$$

The probability distribution is uniform in the remaining $k+1-4=k-3$ states. Therefore if $s_1\geq 2$, and $s_2\geq 2$

$$\lim_{p\to 1}\pi(s_1,s_2)=\frac{1}{k-3}$$

Another important observation about this network is that the steady-state distribution is completed determined by $\pi(k,0)$ and $\pi(k-1,1)$ and a set of recursive equations. Notice that part of the transition matrix for this Markov chain is

	$\pi\left(k,0\right)$	$\pi\left(k-1,1\right)$	$\pi\left(k-2,2\right)$	$\pi\left(k-3,3\right)$
$\pi\left(k,0\right)$	q^2	$2pq$	p^2	0
$\pi\left(k-1,1\right)$	pq^2	q^3+2p^2q	p^3+2pq^2	p^2q
$\pi\left(k-2,2\right)$	p^2q^2	$2p^3q+2pq^3$	$p^4+q^4+4p^2q^2$	$2pq^3+2p^3q$
$\pi\left(k-3,3\right)$	0	p^2q^2	$2p^3q+2pq^3$	$p^4+q^4+4p^2q^2$
$\pi\left(k-4,4\right)$	0	0	p^2q^2	$2p^3q+2pq^3$
$\cdots$	0	0	0	p^2q^2
$\cdots$	0	0	0	0
$\cdots$	0	0	0	0

If we used the notation π_j to indicate a state with j jobs in node one. Then, the balance equations are of the form

$$\pi_0q^2+\pi_1pq^2+\pi_2 2p^2q^2=\pi_0$$

Solving for $\pi\left(k-2,2\right)$, we have that

$$\pi_2p^2q^2=\pi_0\left(1-q^2\right)-\pi_1pq^2$$

hence $\pi\left(k-2,2\right)$ is complete determined by π_0 and π_1. Similarly ,

$$\pi_0 2pq+\pi_1\left(q^3+2p^2q\right)+\pi_2\left(2p^3q+2pq^3\right)+\pi_3p^2q^2=\pi_1$$

solving for $\pi\left(k-3,3\right)$we have that

$$\pi_3p^2q^2=\pi_1\left(1-q^3-2p^2q\right)-\pi_0 2pq-\pi_2\left(2p^3q+2pq^3\right)$$

but $\pi\left(k-2,2\right)$ can be written in terms of π_0 and π_1, therefore $\pi\left(k-3,3\right)$ can also be written in term of π_0 and π_1. In general, the balance equation has the form

$$\pi_{n+2}p^2q^2=\pi_n(1-p^4-q^4-4p^2q^2)-2\pi_{n-1}\left(2p^3q+2pq^3\right)-4\pi_{n-2}p^2q^2.$$

These equations can be solve recursively in terms of π_0 and π_1. Hence , we will concentrate in finding methods to estimate π_0 and π_1 for any k.

We have found the steady-state distribution for different values of k. The following are the probability distributions of π_0 for $k = 4, 5$, and 6. The probability distributions are given here,

$$\pi(4,0) = \frac{-7p + 10p^2 - 7p^3 + 2p^4 + 2}{-47p + 60p^2 - 36p^3 + 8p^4 + 16}$$

$$\pi(5,0) = \frac{-6p + 7p^2 - 4p^3 + p^4 + 2}{-46p + 46p^2 - 22p^3 + 4p^4 + 20}$$

$$\pi(6,0) = \frac{-18p + 37p^2 - 43p^3 + 29p^4 - 11p^5 + 2p^6 + 4}{-178p + 322p^2 - 322p^3 + 188p^4 - 60p^5 + 8p^6 + 48}$$

The figure below shows the graph of these distributions.

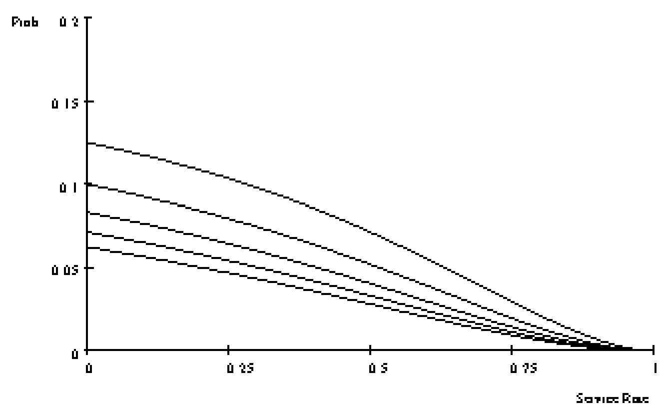

Figure 1: Probabilities for $k = 0$

These sequence of functions appear to be decreasing proportionally and convergent. Using (16), The percentage decrease at $p = 0$ is

$$\frac{1/2(k+1)}{1/2k} = \frac{k}{k+1} \tag{22}$$

hence, we will approximate

$$\pi(k+1,0) \approx \frac{k}{k+1} \cdot \pi(k,0). \tag{23}$$

This approximation is expected to get better as $k \to \infty$. These graphs also show that $\lim_{p \to 1} \pi(k,0) = 0$ as expected based on the conjecture.

Example 12 *We found* $\pi(9,0)$ *by direct computation,*

$$\pi_d(9,0) = \frac{-40p + 96p^2 - 140p^3 + 133p^4 - 84p^5 + 34p^6 - 8p^7 + p^8 + 8}{-584p + 1224p^2 - 1540p^3 + 1274p^4 - 686p^5 + 234p^6 - 46p^7 + 4p^8 + 144}$$

On the other hand , using (22) $\pi_d(9,0)$ *is approximately,*

$$\begin{aligned} & \frac{8}{9}\pi(8,0) \\ = \quad & \frac{8}{9}\frac{-44p + 116p^2 - 185p^3 + 192p^4 - 132p^5 + 58p^6 - 15p^7 + 2p^8 + 8}{-588p + 1368p^2 - 1909p^3 + 1742p^4 - 1033p^5 + 388p^6 - 84p^7 + 8p^8 + 128} \end{aligned}$$

The following figure shows the graphs of the invariant distribution and its approximation (dotted line).

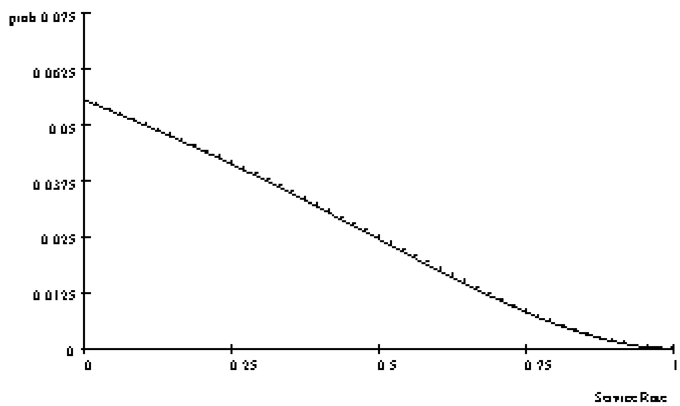

Figure 2 : Probabilites approximation for $k = 8$

This approximation is remarkably good.

It is difficult to find invariant distributions for large k because of the number of calculations involved in finding the inverse of the transition matrix. Most software package can handle up to 20 by 20 matrices.

In general, using the approximation recursively , we get the following expression:

$$\begin{aligned} \pi_0(k) &\approx \frac{k-1}{k}\pi_0(k-1) \\ &= \frac{k-1}{k}\frac{k-2}{k-2}\pi_0(k-2) \\ &= \frac{k-1}{k}\frac{k-2}{k-1}\frac{k-3}{k-2}\cdots\frac{k-r}{k-r+1}\pi(k-r,0) \\ &= \frac{k-r}{k}\pi(k-r,0) \end{aligned}$$

Now we will perform the same kind of analysis for π_1. The following are the distributions of π_1 for k = 4,5, and 6,

$$\pi(4,1) = \frac{-10p + 11p^2 - 6p^3 + p^4 + 4}{-46p + 46p^2 - 22p^3 + 4p^4 + 20}$$

$$\pi(5,1) = \frac{-16p + 31p^2 - 34p^3 + 22p^4 - 8p^5 + p^6 + 4}{-89p + 161p^2 - 161p^3 + 94p^4 - 30p^5 + 4p^6 + 24}$$

$$\pi(6,1) = \frac{-28p + 48p^2 - 47p^3 + 27p^4 - 9p^5 + p^6 + 8}{-176p + 276p^2 - 240p^3 + 122p^4 - 34p^5 + 4p^6 + 56}$$

The figure shows the graphs of the distributions as functions p

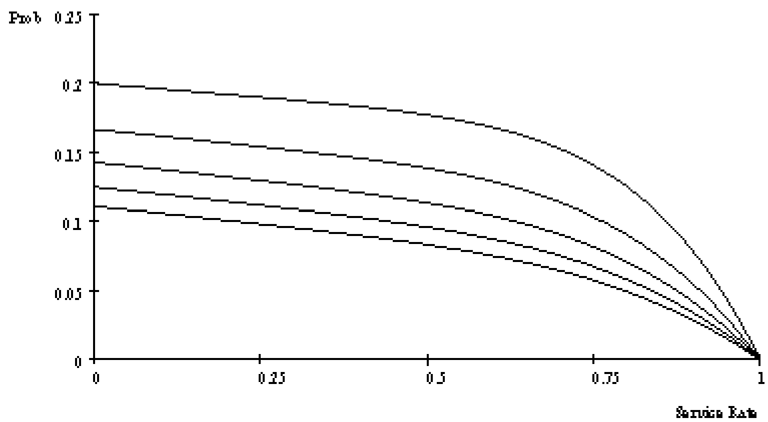

Figure 3 : Probabilities for $k = 1$

These sequence of functions appear to be decreasing proportionally and convergent. Using (23), the percentage decrease is

$$\frac{1/(k+1)}{1/k} = \frac{k}{k+1}$$

hence we will approximate

$$\pi(k,1) \approx \frac{k}{k+1} \cdot \pi(k-1,1).$$

This approximation is expected to get better as $k \to \infty$. The graphs also show that $\lim_{p\to 1} \pi(k-1,1) = 0$ as expected based on the conjecture.

In general , $\pi_1(k)$ will be approximated recursively with the highest known distribution using the following approximation

$$
\begin{aligned}
\pi_1(k) &\approx \frac{k-1}{k}\pi_1(k-1) \\
&= \frac{k-1}{k}\frac{k-2}{k-1}\pi_1(k-2) \\
&= \frac{k-1}{k}\frac{k-2}{k-1}\frac{k-3}{k-2}\cdots\frac{k-j}{k-j+1}\pi(k-j,1) \\
&= \frac{k-j}{k}\pi(k-j,1)
\end{aligned}
$$

Example 13 *Using this method we approximated the distribution for $k = 14$ suing $k = 12$. we have ,*

$$
\begin{aligned}
\pi(14,0) &\approx \frac{13}{14}\frac{12}{13}\cdot\pi(12,0) \approx 0.02593 \\
\pi(13,1) &= \pi(1,13) \approx \frac{13}{14}\frac{12}{13} \approx 0.06192
\end{aligned}
$$

Therefore , using balance equations in terms of π_0 and π_1 we have,

$$
\begin{aligned}
\pi(12,2) &= \pi(2,12) \approx 0.07508 \\
\pi(11,3) &= \pi(3,11) \approx 0.07596 \\
\pi(10,4) &= \pi(4,10) \approx 0.07578 \\
\pi(9,5) &= \pi(5,9) \approx 0.075870 \\
\pi(8,6) &= \pi(6,8) \approx 0.07586 \\
\pi(7,7) &= \pi(7,7) \approx 0.07580
\end{aligned}
$$

Exercise 14 *notice that $\sum \pi_i = 2 \cdot 0.466\,43 + 7.5870 \times 10^{-2} = 1.008 \approx 1.0$*

Another way that can be use to check if the approximations are acceptable is by using them in calculating the expected number of customer at a given node which is a know quantity.

Lemma 15 *For a closed cyclic network with n nodes , $r = 2$ servers per node , k jobs and constant service rate p, the expected number of jobs at the first node is $\frac{k}{2}$*

Proof. Do to the symmetry $\pi(j,k-j) = \pi(k-j,j)$ and

$$j\pi(j,k-j) + (k-j)\pi(k-j,j) = k\pi(j,j-k)$$

so

$$E[\text{Jobs at the first node}] \tag{24}$$

$$= \sum_{j=0}^{k} j\pi(j,k-j)$$

If k is odd the above expression is equal

$$= k\sum_{j=0}^{k/2} \pi(j,j-i) = \frac{k}{2}$$

If k is even, (4.15) equals

$$k\sum_{j=0}^{k/2} \pi(j,k-j) + \frac{k}{2}\pi\left(\frac{k}{2},\frac{k}{2}\right)$$

$$= \frac{k}{2}$$

■

In the example 37 , using the approximation results,

$$E[\text{jobs at node one}]$$

$$= 14 * (0.026 + 0.062 + 0.075 + 0.076 + 0.076 + 0.076 + 0.076) + 7 * 0.076$$

$$= 7.07 \approx \frac{k}{2}$$

Conclusion 16 *Finding equilibrium vectors for discrete-time queueing networks is more difficult because events can happen simultaneously in an interval of time. However, in some cases, discrete distributions can be written in term of continuous-time distributions. Continous-time invariant vectors are well known in the literature. They are use to study the network performance measures. Therefore, it may be possible to investigate performance indicators of discrete-time networks using the fact that in some cases discrete-time distributions are perturbations of the continuous-time distributions for the equivalent network.*

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