# A Few Practical Formulas for the Design of Reinforced Concrete Flexural Members

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**ABSTRACT**

In this paper, a series of formulas corresponding to the so-called Unified Design Method (UDM) characterizations of *tension-controlled*, *compression-controlled*, and, *transition-zone* beam sections are presented. The formulas are in terms of reinforcement ratios and completely avoid the use of strain limits and/or $c/d_t$ ratios. Many of the formulas presented here are well known and established, but a few are new. In particular, a formula for the compression-controlled reinforcement ratio limit is introduced and a direct design procedure for transition-zone sections is developed and described. It is hoped that the formulas and ideas presented in this paper will prove useful both for the instructor in the classroom, and for the practicing structural engineer. Somewhat detailed derivations of most of the formulas in the paper are presented and several numerical examples to illustrate their use are provided at the end of the paper.

**Keywords:** Reinforced concrete; Unified Design Method; reinforcement ratio limits; compression-controlled sections; transition-zone sections; tension-controlled sections.

## 1. INTRODUCTION

In 2002 the American Concrete Institute (ACI) introduced a new approach to the design of structural members subject to bending. This change was introduced through the ACI-318-02 publication (ACI 2002). The motivation for the change was to achieve uniformity between the design procedures used for the design of prestressed, reinforced, and compression members (Ghosh 2004). As a consequence of this, the new approach to design has been termed unified design method (UDM) by some authors (Munshi 1998, Hassoun 2002). We will adopt this name here. The UDM introduced the ideas of *tension-controlled*, *compression-controlled*, and, *transition-zone* sections. This characterization of reinforced concrete sections requires consideration of strain limits and $c/d_t$ ratios. The traditional approach to design of reinforced concrete sections relies instead on the concept of reinforcement ratios. In the author's opinion, the concept of reinforcement ratios is simpler, more intuitive, and has more pedagogical appeal than the strain limits concept. The traditional approach to design can still be used (as of the ACI-318-2008 edition) but it has been relegated to an appendix of the code and it is likely to disappear from it in subsequent editions (ACI 2008). The objective of this paper is to introduce a series of formulas for the UDM that will allow the design and analysis of flexural members using reinforcement ratio limits.

## 2. TENSION-CONTROLLED, TRANSITION-ZONE, AND COMPRESSION-CONTROLLED SECTIONS

Fig. 1 shows the usual representation of the flexural stresses and strains in a typical reinforced concrete section. The notation and nomenclature are the usual, with $c$ representing the depth of the neutral axis of the section; $d$, representing the distance from the top of the section to the centroid of the main reinforcement; $d_t$ the distance from the top of the beam to the bottom layer of the tensile reinforcement; $a$, the depth of the compression block; $A_s$, the area of the main reinforcement; and $f_s$, the stress in the tension steel.

Referring to Fig. 1b, we observe that,

$$\frac{c}{d_t} = \frac{\varepsilon_{cu}}{\varepsilon_{cu} + \varepsilon_t} \tag{1}$$

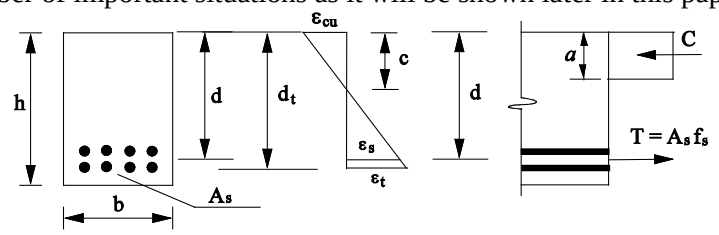
On the other hand, from the stress diagram (Fig. 1c), we get the familiar expression for the depth of the compression block *a*, as:

$$a = \frac{A_s f_s}{0.85 f_c^{'} b} \tag{2}$$

Comparing (1) and (2), recalling that $a = \beta_1 c$ and that the reinforcement ratio is defined as $\rho \equiv \frac{A_s}{bd}$, we conclude that,

$$\rho = \frac{0.85\beta_1 f_c^{'}}{f_s}\left(\frac{\varepsilon_{cu}}{\varepsilon_{cu} + \varepsilon_t}\right)\frac{d_t}{d} \tag{3}$$

Eq. (3) is a completely general relationship for singly reinforced concrete beam sections. This equation can be specialized to fit a number of important situations as it will be shown later in this paper.




**a)** Section **b)** Strains **c)** Stresses

**Figure 1. Schematic Representation of Stresses and Strains in a Typical Reinforced Concrete Section.**

### 2.1 Tension-controlled Sections

The ACI-318-02 defines a tension-controlled section as a section such that the strain $\varepsilon_t$ in the lowermost layer of steel is greater than or equal to 0.005. This is to ensure that the main steel yields well before the concrete crushes, providing enough ductility for the section even for seismic zones. This definition is inspired by the yield strain of grade 420 steel (G420 for short) which is approximately equal to 0.002 (The definition of tension-controlled section is the same for other grades of steel despite the fact that $\varepsilon_y$ is different: for G520 steel for instance, $\varepsilon_y$ = 0.0026). Now, by substituting $\varepsilon_{cu}$ = 0.003 and $\varepsilon_t$ = 0.005 in (1), one obtains,

$$\frac{c}{d_t} = 0.375 \tag{4}$$

Eq. (4) is the most commonly used equation to test for tension-controlled sections in the literature (see e.g., Nawy 2002). It is the purpose of this paper to replace expressions of the type (4) with equivalent expressions in terms of the reinforcement ratio $\rho$. To obtain the tension-controlled limit, it suffices to substitute $\varepsilon_t$ = 0.005 and $f_s = f_y$ in (3). The expression for the tension-controlled reinforcement ratio is then obtained as:

$$\rho_{tcl} = \frac{2.55\beta_1 f_c^{'}}{8 f_y}\frac{d_t}{d} \tag{5}$$

Eqn. (5) assumes that the steel at a depth *d* from the top of the beam (i.e., at the level of the centroid of the main reinforcement) has also yielded when $\varepsilon_t$ = 0.005[1].

### 2.2 Compression-controlled Sections and Transition-Zone Sections

Beam sections for which $\varepsilon_t \le \varepsilon_y$ are classified as compression-controlled sections and are not permitted by the ACI code as of the 2002 edition. Occasionally however, it might still be required to find the moment carrying capacity of one of these sections since it may have been designed in a non-compliant way by mistake, or using a

[1] There is a slight possibility that this will not happen. The steel at the level of the main reinforcement will not yield (despite the fact that $\varepsilon_t$ = 0.005) in the very unusual circumstance that the ratio $d/d_t$ is less than $\frac{f_y + E_s\varepsilon_{cu}}{E_s\varepsilon_{cu} + 0.005E_s}$. This value is approximately equal to 0.6375 (0.6336) for G420 (G60) steel.

different standard. Or the section may have been designed during the period from 1999 to 2002 in which the ACI code was not clear about whether these sections were allowed or not (see McGregor and Wight 2005, page 127). The strength reduction factor for compression-controlled sections (with ties) is $\phi = 0.65$.

Beam sections for which $$\varepsilon_y \le \varepsilon_t \le 0.005 \quad (6)$$

are considered transition-zone sections. For transition-zone sections with ties, the strength reduction factor $\phi$ must be linearly interpolated between the compression-controlled limit value of 0.65 and the tension-controlled limit value of 0.9, as given by the next equation (for G420 steel only): $$\phi = 0.65 + \frac{250}{3}(\varepsilon_t - 0.002) \quad (7)$$

Now, the compression-controlled limit corresponds to $\varepsilon_t = \varepsilon_y$. By substituting $\varepsilon_t = \varepsilon_y \equiv \frac{f_y}{E_s}$ in Eq. (1), we obtain,

$$\frac{c}{d_t} = \frac{E_s \varepsilon_{cu}}{E_s \varepsilon_{cu} + f_y} \quad (8)$$

which upon substitution of the values $\varepsilon_{cu} = 0.003$ (Nawy 2002), and $E_s = 200{,}000$ MPa (29,000 ksi), becomes[2],

$$\frac{c}{d_t} = \frac{600}{600 + f_y} \quad (9)$$

in metric units, and

$$\frac{c}{d_t} = \frac{87}{87 + f_y} \quad (10)$$

in US customary units.

For G420 (G60) steel the compression-controlled limit corresponds to $\varepsilon_y = 0.00210(0.0207)$. By substituting $\varepsilon_y = 0.00210$ (0.0207) and $\varepsilon_{cu} = 0.003$ in Eq. (1), we obtain, $$\frac{c}{d_t} = 0.5882 \quad (11)$$

in metric units, and

$$\frac{c}{d_t} = 0.5918 \quad (12)$$

in US customary units.

If $\varepsilon_t = \varepsilon_y$ is approximated as 0.002, either of the previous two formulas can be approximated as:

$$\frac{c}{d_t} = \frac{3}{5} = 0.6 \quad (13)$$

Eq. (13) is the expression most commonly used to test for compression-controlled sections in the literature (see e.g., Nawy 2002). The linear interpolation formula for $\phi$ (Eq. (7)) can also be written in terms of the $c/d_t$ ratio as follows (for G420 steel):

$$\phi = 0.65 + 0.25\left(\frac{d_t}{c} - \frac{5}{3}\right) \quad (14)$$

It is also useful to write this formula in terms of the depth of the compression block *a*, as,

$$\phi = \frac{0.7}{3} + \frac{0.25\beta_1 d_t}{a} \quad (15)$$

It can be concluded now that transition-zone sections are characterized by,

$$0.375 \le \frac{c}{d_t} \le 0.6 \quad (16)$$

[2] These formulas are unit dependent. For the metric formula, stresses must be in MPa. For the US customary units formula, stresses must be in ksi.

Eqs. (16) is valid for G420 (G60) steel only. A more general, and slightly more accurate expression to characterize transition-zone sections can be obtained by using Eq. (8) instead of Eq. (13) as the upper limit in (16) as follows,

$$\frac{3}{8} \le \frac{c}{d_t} \le \frac{E_s \varepsilon_{cu}}{E_s \varepsilon_{cu} + f_y} \tag{17}$$

This formula is of course general and *unit independent*. In what follows, only general formulas will be given. To specialize any of the these formulas either to the US customary system or the metric system, it suffices to replace the term $E_s \varepsilon_{cu}$ with the quantity ``87" for the US customary system and with ``600" for the metric system.

In terms of strains, the expression that characterizes transition-zone sections for G420 (G60) steel becomes,

$$0.002 \le \varepsilon_t \le 0.005 \tag{18}$$

To ensure enough ductility however, the ACI code further restricts the value of $\varepsilon_t$ to a *minimum* value of 0.004. When this value is substituted in Eq. (1), it can be concluded that the value of the *maximum* allowable $c/d_t$ ratio is,

$$\frac{c}{d_t} = \frac{3}{7} \approx 0.4286 \tag{19}$$

which means that the range of *permitted* (or allowed) transition-zone sections is actually,

$$0.375 \le \frac{c}{d_t} \le 0.4286 \tag{20}$$

which is of course a subinterval of (16).

When $c/d_t = 3/7$ is used in conjunction with Eq. (14), the corresponding range of permitted $\phi$ values is obtained as,

$$0.8167 \le \phi \le 0.9 \tag{21}$$

When $\varepsilon_t = 0.004$, together with the value of $\varepsilon_{cu} = 0.003$, and $f_s = f_y$, are substituted into (3), an expression for the maximum permitted value of the reinforcement ratio is obtained as:

$$\rho^*_{\max} = \frac{2.55 \beta_1 f_c^{'}}{7 f_y} \frac{d_t}{d} \tag{22}$$

where an asterisk is used to distinguish this ratio from the traditional maximum reinforcement ratio ($\rho_{\max} = 0.75 \rho_b$ (ACI 2008, Nawy 2002)). The permitted range of transition-zone sections, in terms of reinforcement ratios, can now be written as,

$$\rho_{tcl} \le \rho \le \rho^*_{\max} \tag{23}$$

which is completely equivalent to Eq. (21).

Now, by comparing Eq. (5) with Eq. (22), it can be concluded that,

$$\rho^*_{\max} = \frac{8}{7} \rho_{tcl} \tag{24}$$

It is interesting to compare Eq. (24) with the traditional expression for the balanced reinforcement ratio. When this is done, the following relationship is obtained,

$$\rho^*_{\max} = \frac{3}{7} \left( \frac{E_s \varepsilon_{cu} + f_y}{E_s \varepsilon_{cu}} \right) \frac{d_t}{d} \rho_b \tag{25}$$

where $\rho_b$ is the traditional balanced reinforcement ratio given by,

$$\rho_b = \frac{0.85 \beta_1 f_c^{'}}{f_y} \left( \frac{E_s \varepsilon_{cu}}{E_s \varepsilon_{cu} + f_y} \right) \tag{26}$$

For G420 (G60) steel and for the particular case when $d_t = d$ (one layer of main reinforcement steel), Eq. (29) becomes,

$$\rho^*_{\max} \approx 0.73 \rho_b \tag{27}$$

which compares very well with the traditional value of the maximum reinforcement ratio (ACI 2008, Nawy 2002),

$$\rho_{\max} = 0.75\rho_b \tag{28}$$

### 2.2.1 RELATIONSHIP BETWEEN $\varepsilon_s$ AND $\varepsilon_t$

In order to find an expression for the reinforcement ratio that corresponds to the compression- controlled limit, it is necessary to first establish a relationship between the strain in the lowermost layer of steel $\varepsilon_t$ and the strain at the centroid of the main steel reinforcement $\varepsilon_s$. The reason for this is that when the lowermost steel is exactly at the yield stress (compression-controlled limit), the stress at the centroid of the main steel reinforcement is less than the yield stress.

Referring to Fig. 1b, and using similar triangles, we obtain,

$$\varepsilon_s = \frac{d-c}{d_t - c}\varepsilon_t \tag{29}$$

or, in terms of stresses,

$$f_s = \frac{d-c}{d_t - c} f_t \tag{30}$$

When the lowermost reinforcement is exactly at the yield point, the stress at the centroid of the main reinforcement is then,

$$f_{sccl} = \frac{d-c}{d_t - c} f_y \tag{31}$$

Now, making use of Eq. (1), and multiplying numerator and denominator of (37) by $E_s$, we get,

$$f_{sccl} = \frac{f_y d - E_s \varepsilon_{cu}(d_t - d)}{d_t} \tag{32}$$

Substituting Eq. (32) into Eq. (3), and multiplying and dividing again by $E_s$, we finally obtain,

$$\rho_{ccl} = \frac{0.85\beta_1 f_c^{'} d_t}{f_y d - E_s \varepsilon_{cu}(d_t - d)}\left(\frac{E_s \varepsilon_{cu}}{E_s \varepsilon_{cu} + f_y}\right)\frac{d_t}{d} \tag{33}$$

In Eqs. (31) to (33) the subindex *ccl* stands for compression-controlled.

The range of transition-zone sections can now be written exclusively in terms of reinforcement ratio limits as,

$$\rho_{tcl} \le \rho \le \rho_{ccl} \qquad (34)\backslash$$

which is completely equivalent to (17).

Note that when $d_t = d$, the expression for the compression-controlled limit (Eq. (33)) reduces to the expression for the traditional balanced reinforcement ratio (Eq. (26)).

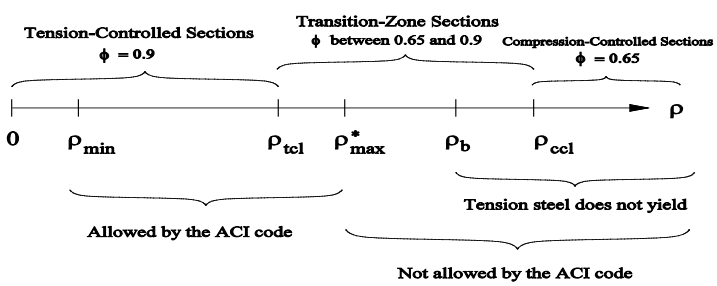



**Figure 2. Schematic Representation of the Relative Values of the Reinforcement Limits Presented in this Paper. In this figure $\rho_{\min}$ represents the minimum reinforcement ratio for beams given by: $\rho_{\min} = 0.25\sqrt{f_c^{'}} / f_y (\ge 1.4 / f_y)$ (see e.g., Nawy 2002).**

Fig. 2 illustrates schematically the relative values of all reinforcement limits together with their meaning in terms of the behavior of the corresponding sections. It is important to note (as it can be seen in Fig. 2) that for the

normally occurring case when $d_t > d$, the value of $\rho_{ccl}$ is larger than $\rho_b$. This means that for many transition-zone sections, the main tensile steel reinforcement does not yield. It is then necessary to solve a quadratic equation for *a* or for $f_s$ to find the moment capacity of the sections. The following is the quadratic equation for *a* (see e.g. McGregor and Wigth 2005):

$$(0.85 f_c^{'})a^2 + (E_s \varepsilon_{cu} d\rho)a - E_s \varepsilon_{cu} \rho \beta_1 d^2 = 0 \tag{35}$$

## 3. Design Aspects

Most authors use a trial and error procedure for the design of reinforced concrete sections (see e.g., Nawy 2002, McGregor and Wight 2005). The author prefers to use a quadratic formula for the reinforcement ratio $\rho$. This formula is derived next. Similar formulas can be found in the literature (see e.g., Hassoun 2002, page 100). We start with the relationship for the nominal moment carrying capacity of a singly reinforced section with the tension steel yielding,

$$M_n = A_s f_y (d - \frac{a}{2}) \tag{36}$$

or, by using Eq. (2) with $f_s = f_y$, and multiplying both sides of (51) by $\phi$,

$$\phi M_n = \phi A_s f_y (d - \frac{A_s f_y}{1.7 f_c^{'} b}) \tag{37}$$

Recognizing now that the most economical design results when $\phi M_n = M_u$, substituting $\rho bd$ for $A_s$ in Eq. (37), and reorganizing terms, we get the following quadratic equation for $\rho$,

$$f_y \rho^2 - 1.7 f_c^{'} \rho + \frac{1.7 M_u f_c^{'}}{\phi b d^2 f_y} = 0 \tag{38}$$

whose solution is:

$$\rho = \frac{0.85 f_c^{'} - \sqrt{(0.85 f_c^{'})^2 - \frac{1.7 f_c^{'} M_u}{\phi b d^2}}}{f_y} \tag{39}$$

The minus sign before the square root has been chosen in (39) to obtain a reinforcement ratio that is less than one. Eq. (39) is very useful as long as $\phi$ is equal to 0.9 (i.e., for tension-controlled sections). If $0.8167 \le \phi < 0.9$ (i.e., transition-zone sections), a different procedure is necessary. This procedure is described next.

### 3.1 Design of Transition-Zone Sections

For transition-zone sections, the value of $\phi$ must be interpolated using Eq. (14) or Eq. (15). This means that the value of $\phi$ in Eq. (39) depends implicitly on $\rho$. For these sections, it is still possible to use Eq. (39) as a first approximation, and then iterate until a sufficiently accurate result is obtained for $\rho$ and $\phi$. Special design aids that rely on a series of charts have also been devised for this case (Munshi 1998, 1999). Aschheim et al. (2008) also present an iterative procedure to design transition-zone sections. An exact and direct approach that does not require iterations is presented next.

#### 3.1.1 Direct Procedure for the Design of Transition-Zone Sections

It turns out that it is possible to derive an equation that can be solved directly for $\rho$ to obtain the exact combination of $\rho$ and $\phi$ for transition-zone sections. Remarkably enough, in spite of the apparent high nonlinearity of the relationship between $\rho$ and $\phi$, the resulting equation is also a *quadratic* equation. The first step is to use Eq. (2) in conjunction with Eq. (15) and the definition of $\rho$, to write $\phi$ in terms of $\rho$ as:

$$\phi = \frac{0.7}{3} + \frac{0.85 \beta_1 f_c^{'} d_t}{4 \rho f_y d} \tag{40}$$

Substituting this value into Eq. (45) we get, after some algebraic manipulation,

$$2.8 f_y^2 b d^2 \rho^3 + (2.55 \beta_1 f_c^{'} f_y b d d_t - 4.76 f_c^{'} f_y b d^2)\rho^2 + (20.4 M_u - 4.335 \beta_1 f_c^{'} b d d_t) f_c^{'} \rho = 0 \tag{41}$$

This is a cubic equation in $\rho$ with the trivial solution $\rho = 0$. We are then left with the following *quadratic* equation to solve for the nontrivial reinforcement ratio $\rho$.

$$\rho^2 - \left( \frac{1.7 f_c^{'}}{f_y} - \frac{2.55 \beta_1 f_c^{'} d_t}{2.8 f_y d} \right) \rho + \left( \frac{5.1 f_c^{'} M_u}{0.7 b d^2 f_y^2} - \frac{1.08375 \beta_1 (f_c^{'})^2 d_t}{0.7 f_y^2 d} \right) = 0 \qquad (42)$$

The value of $\rho$ found from this equation provides the design moment $\phi M_n = M_u$ that corresponds to the exact interpolated value of $\phi$ given by Eq. (15) (see Example 2 below).

## 4. Analysis and Design of T- and Doubly-Reinforced Sections

With only minor modifications, the formulas that are used to analyze and/or design singly reinforced sections can be used to analyze and/or design T- and doubly-reinforced sections.

### 4.1 T-Sections

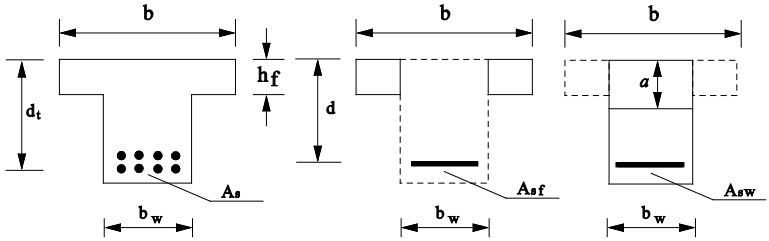
When dealing with T-sections, it is customary to idealize the actual T-section as two separate sections that, when superimposed, reproduce the original section. These two sections are traditionally called the F-beam (corresponding to the flanges) and the W-beam (corresponding to the web) (Fig. 3). Within this framework, the area of main reinforcement steel that is balanced exactly by the compressive forces on the flanges is denoted by $A_{sf}$. The area of reinforcement steel that corresponds to the W-beam is then given by, $A_{sw} = A_s - A_{sf}$ (43)

where $A_s$ is the total area of main reinforcement of the section.

It is convenient to define reinforcement ratios corresponding to the F-beam and to the W-beam as $\rho_f \equiv \frac{A_{sf}}{b_w d}$, and $\rho_w \equiv \frac{A_{sw}}{b_w d}$. Note that in both of these equations, the denominator involves the web width $b_w$. This ensures that one can write,

$$\rho = \rho_w + \rho_f \qquad (44)$$

where $\rho = \frac{A_s}{b_w d}$ is the total reinforcement ratio of the T-section.




a) Section b) F-beam c) W-beam

**Figure 3. Schematic Representation of a Typical T-beam Reinforced Concrete Section.**

To apply the formulas of reinforcment ratios presented above to the analysis and design of T-sections, it suffices to keep in mind that the quantity that must be compared with the different reinforcement limits is $\rho_w$ (since this is the ratio that corresponds to the rectangular portion of the section). For instance, if $\rho_w \le \rho_{tcl}$, the section is tension-controlled; if $\rho_{tcl} \le \rho_w \le \rho_{ccl}$ the section is in the transition-zone, etc.

The remaining equations for the analysis and design of T-sections are standard and can be found for instance in McGregor and Wight 2005.

### 4.2 Doubly-Reinforced Sections

The typical notation and nomenclature for doubly-reinforced sections is shown in Fig. 4. In analogy with T-sections, doubly-reinforced sections are also considered to be composed of two parts that superimposed reproduce the original section. These two parts can be called Beam-1 and Beam-2. Beam-1 is composed of the compression

reinforcement plus the portion of the tensile reinforcement that exactly balances the forces in the compression reinforcement. Beam-2 includes the compression block on the concrete and the portion of the tension reinforcement that balances the compressive forces on the concrete (Fig. 4).

The only difficulty with the analysis and design of doubly-reinforced sections is the fact that sometimes the compression reinforcement does not yield at failure. When this is the case, it is necessary to determine the value of the stress $f_s^{'}$ in the compression reinforcement at failure. In addition, it is necessary to determine when the compression reinforcement yields and when it does not. This depends on the amount of tension reinforcement in the section (see e.g., McGregor and Wight 2005). All this is done using the procedure outlined below.

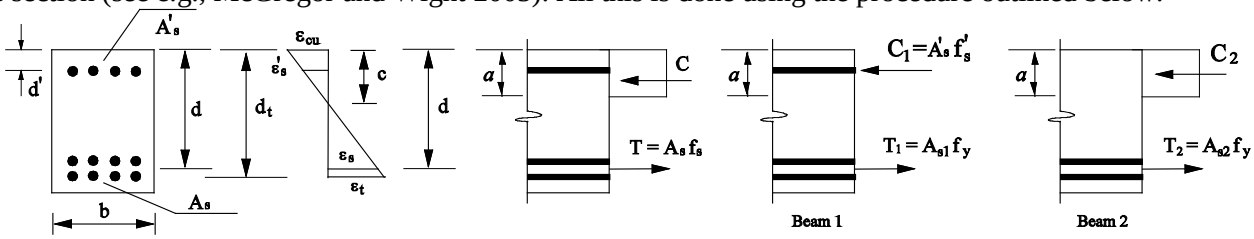



**Figure 4. Stresses and Strains in a Typical Doubly-Reinforced Concrete Section.**

The areas of steel reinforcement corresponding to the original beam section and to Beams 1 and 2 are related by $A_s = A_{s1} + A_{s2}$. Now, by the definition of Beam-1 (see Fig. 4**),** $A_{s1} = \dfrac{A_s^{'} f_s^{'}}{f_y}$ and

therefore $A_{s2} = A_s - \dfrac{A_s^{'} f_s^{'}}{f_y}$. It is useful to define also, $\rho^{'} = \dfrac{A_s^{'}}{bd}$, and write, $\rho_2 = \rho - \rho^{'} \dfrac{f_s^{'}}{f_y}$ (45)

where $\rho = \dfrac{A_s}{b_w d}$, as usual.

It can be shown (by a procedure similar to the one outlined in Section 2.), that the limiting reinforcement ratio to determine whether the compression reinforcement has yielded at failure is (see e.g., McGregor and Wight 2005).

$$\rho_{cry} = \frac{0.85\beta_1 f_c^{'}}{f_y}\left(\frac{E_s\varepsilon_{cu}}{E_s\varepsilon_{cu} - f_y}\right)\frac{d^{'}}{d} \tag{46}$$

where the subindex *cry* stands for ``compression reinforcement yielding". Now, it is important to emphasize that the quantity that must be compared with $\rho_{cry}$ (and with any other reinforcement ratio limit for that matter) to perform this test is $\rho_2$. The difficulty with this approach is of course that to be able to calculate $\rho_2$, $f_s^{'}$ must be known (see Eq. (45)). A way around this problem is to approximate $\rho_2$ as $\tilde{\rho}_2 \approx \rho - \rho^{'}$, and perform the test using $\tilde{\rho}_2$ instead. This approximation does not pose any problems since $\rho_2 \geq \tilde{\rho}_2$. The test can then be performed as follows: if $\tilde{\rho}_2 > \rho_{cry}$, the compression reinforcement *yields*. This of course means that $f_s^{'} = f_y$; if $\tilde{\rho}_2 < \rho_{cry}$, the compression reinforcement *does not yield*. This means that $f_s^{'} < f_y$, and the stress in the compression reinforcement $f_s^{'}$, is unknown. When the compression reinforcement yields, the nominal moment carrying capacity of a doubly-reinforced section can be determined by simply adding the contributions from Beam-1 and Beam-2 in the standard way. When the compression reinforcement does not yield, it is necessary to determine *a* (the depth of the compression block) by means of the following quadratic equation (see e.g., McGregor and Wight 2005):

$$(0.85 f_c^{'} b)a^2 - (A_s f_y - A_s^{'} E_s \varepsilon_{cu})a - A_s^{'} E_s \varepsilon_{cu} \beta_1 d^{'} = 0 \tag{47}$$

Once Eq. (55) is solved for *a*, the stress in the compression steel can be found as:

$$f_s^{'} = E_s \varepsilon_{cu}\left(1 - \frac{\beta_1 d^{'}}{a}\right) \tag{48}$$

The nominal moment carrying capacity of a doubly-reinforced section can then be determined in the standard way. Alternatively, when the compression reinforcement does not yield, it can be shown that the stress in the compression steel can be found as:

$$f_s^{'} = (\frac{E_s \varepsilon_{cu}}{2} + \frac{\rho f_y}{2\rho^{'}}) - \sqrt{(\frac{E_s \varepsilon_{cu}}{2} + \frac{\rho f_y}{2\rho^{'}})^2 - E_s \varepsilon_{cu}(\frac{\rho f_y}{\rho^{'}} - \frac{0.85\beta_1 f_c^{'} d^{'}}{\rho^{'} d})} \tag{49}$$

It should be emphasized that when classifying, analyzing, or designing doubly-reinforced sections, the quantity that must be compared with the reinforcement limits is $\rho_2$ (since this is the ratio that corresponds to the rectangular portion of the section). For instance, if $\rho_2 \le \rho_{tcl}$ the section is tension-controlled. if $\rho_{tcl} \le \rho_2 \le \rho_{ccl}$ the section is in the transition zone, etc.

## 5. NUMERICAL EXAMPLES

The following numerical examples illustrate the use of some of the formulas and concepts presented above. Example 1 illustrates the use of Eq. (33); and example 2, that of Eq. (42).

### 5.1 EXAMPLE 1

The purpose of this example is to illustrate the use of Eq. (33) and to demonstrate its accuracy. To do this, we will show that for a beam with $\rho = \rho_{ccl}$, the ratio $c/d_t$ is exactly equal to 0.5882 as indicated by Eq. (11). The compression controlled reinforcement ratio is given by Eq. (33). For the dimensions of the beam shown in Fig. 5a, and assuming G420 steel and $f_c^{'}$= 25 MPa (3,617 psi), Eq. (33) yields: $\rho_{ccl} = 0.03712$. Now, the balanced reinforcement ratio (Eq. (26)) is: $\rho_b = 0.0253$. On the other hand, the reinforcement ratio for the beam shown in Fig. 5a is $\rho = 0.03718$ (area of a #32M bar: 819 mm$^2$; area of a #29M bar: 645 mm$^2$). In other words, the reinforcement of the beam shown in Fig. 5a corresponds, in practical terms, to the compression controlled limit. In addition, since this reinforcement ratio is larger than the balanced reinforcement ratio ($\rho > \rho_b$), the tension reinforcement does not yield at failure. Therefore, to find the value of the depth of the compression block it is necessary to solve Eq. (35) for *a*, as illustrated below.

For the material properties given above, and the dimensions and parameters of the beam shown in Fig. 5a, Eq. (35) yields: $21.25a^2 + 10{,}038.6a - 3{,}839{,}764.5 = 0$. The solution of this equation is *a* = 250.0963 mm (9.45 in). Therefore, $c = \frac{a}{\beta_1} = 294.231$ mm (11.58 in). Therefore: $\frac{c}{d_t} = 0.5885$, which corresponds almost exactly to the compression controlled limit of 0.5882 given by Eq. (11).

### 5.2 EXAMPLE 2

The purpose of this example is to illustrate the use of Eq. (42) for the design of transition-zone sections and to verify its validity. Suppose that it is necessary to find the reinforcement for a section with the dimensions shown in Fig. 5b, for the following data: $M_u$=396 kN.m (291.4 kips-ft); $f_c^{'}$ =25 MPa (3,617 psi), and $f_y$=420 MPa (60.77 ksi). A quick check, using Eq. (39) (with $\phi$=0.9) yields $\rho \approx 0.018$. On the other hand, the tension-controlled, and maximum reinforcement ratios are, respectively: $\rho_{tcl} = 0.01792$ and $\rho^*_{\max} = 0.02048$. Therefore, the section is very likely in the transition-zone and $\phi$ is less than 0.9. Using now Eq. (42) to find $\rho$, the following quadratic equation is obtained: $\rho^2 - 0.05\rho + 0.0005885 = 0$, which yields: $\rho = 0.01896 \approx 0.019$. This corresponds to an area of steel $A_s$ = 2992.5 mm$^2$ (4.64 in$^2$). For this area of

steel, the values of $a$, $\phi$, and $M_n$, are as follows (using Eqs. (2), (15), and (36)) $a = 168.988$ mm (6.65 in$^2$); $\phi = 0.8621$; $M_n = 459.39$ kN.m (338 kips-ft); which corresponds exactly to: $\phi M_n = 396.04$ kN.m (291.4 kips-fr) $\approx M_u$. The design can be completed by choosing 3 # 29M bars plus 3 # 22M bars to yield a total area of steel $A_s = 3096$ mm$^2$ (4.8 in$^2$) (area of a #29M bar: 645 mm$^2$; area of a #22M bar: 387 mm$^2$) and a reinforcement ratio $\rho = 0.01966$ ($< \rho^*_{\max}$). The final design is shown in Fig. 5b.

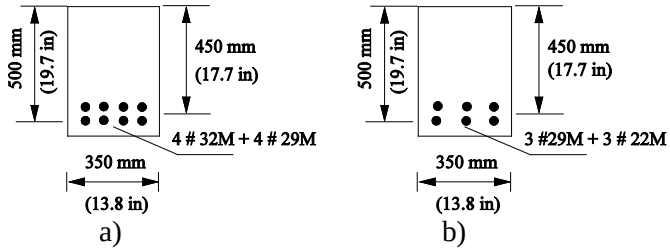



**Figure 5. Beam Sections For Numerical Examples 1 and 2.**

## 6. CONCLUDING REMARKS

A formulation of the Unified Design Method (UDM) for the anslysis and design of reinforced concrete flexural members has been presented. The formulation is based entirely on the concept of *reinforcement ratios*. This is in contrast to the Unified Design Method which relies heavily on strain limits and $c/d_t$ ratios. The concept of reinforcement ratios is not only simpler and more intuitive but in addition, it allows for the teaching of reinforced concrete much in the same way as it has been traditionally done. Many of the formulas presented are well-known but some are not found in the literature. In particular, the expression for the compression-controlled reinforcement ratio, and the procedure for the design of transition-zone sections presented in this paper are new.

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