Modelling the Ball-and-Beam System From Newtonian Mechanics and from Lagrange Methods

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ABSTRACT

The Ball and Beam is a system commonly used to expose undergraduate students to controller design. One important step of this design process is to develop a mathematical model to describe the behavior of the sytem. There are several possible methods for deriving the equations of motion of a dynamical system. These include Langrangian methods and Newtonian mechanics. Many authors fail to adequately derive the model for the Ball-and-Beam system because they obviate several acceleration terms. Although these terms do not affect the linear model, they are important for nonlinear simulations. When the model is derived from Euler-Lagrange methods, these terms appear naturally. In this paper it is demonstrated that the equations of motion obtained from both methods are identical. From the equation of motion, nonlinear state-space equations are developed. The nonlinear equations are then linearized about the equilibrium point and a transfer function suitable for a linear controller design is obtained.

Keywords: Ball and Beam System, Dynamical Systems, Langrangian Mechanics, Newtonian Mechanics, System Modeling

1. INTRODUCTION

System modeling is an extremely important part of the control system design process. An accurate model of a dynamical system allows us to better understand the physical system and facilitates the analysis and design of controllers. The behavior of dynamical systems are described by differential equations. One of the first steps in the design process is to derive these equations from physics laws. Seveal methods are available when deriving the dynamic equations. For mechanical systems, two common approaches are Lagrangian Mechanics and Newtonian Mechanics. While both of these methods yield identical results for suitable systems, one of the two methods may be significantly simpler, depending on the nature of the system. Many authors have derived the equations for the Ball and Beam system using the Lagrangian method (Hauser et al. 1992), (Hirsch, 1989). Others have derived these equations using Newtonian mechanics, but have failed to consider the effect of the rotating coordinate system, leading to missing terms in the equations (Virseda, 2004), (Hamed, 2010). The purpose of this paper is to provide a detailed derivation of the Ball and Beam dynamical equations using Newtonian mechanics and demonstrate that they are identical to the results obtained by the Lagrangian method. Nonlinear and linearized state-variable models for the Ball and Beam system are derived from the equations of motion.

2. MODEL FROM LAGRANGIAN MECHANICS

The Langrange method is an energy based approach for deriving the equations of motion of a dynamical system. This is convenient since it does not require the use of vectors. For this reason, many complicated systems are often analyzed using this method. We will now derive the equations of motions for the Ball and Beam using this method. Consider the Ball and Beam system in Figure 1 (Hirsch, 1989).

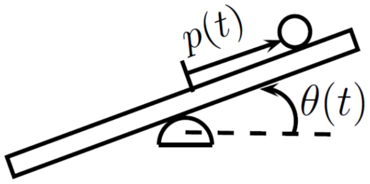

Figure 1: Ball and Beam System

The ball rolls on the beam without slipping under the action of the force of gravity. The beam is tilted from an external torque to control the position of the ball on the beam. We first define a set of generalized coordinates which fully describe the system. The generalized coordinates are defined as

$$q(t) = \begin{bmatrix} p(t) \\ \theta(t) \end{bmatrix} \tag{1}$$

where *p(t)* is the position of the ball on the beam and *θ(t)* is the angle of the beam. The Lagrangian of a system is a quantity which is defined as (D'Souza, 1984)

$$L = K - U \tag{2}$$

where *K* is the kinetic energy and *U* is the potential energy of the system. To facilitate the evaluation of *K* and *U*, we define the Cartesian coordinates *x(t) and y(t)* as shown in Figure 2.

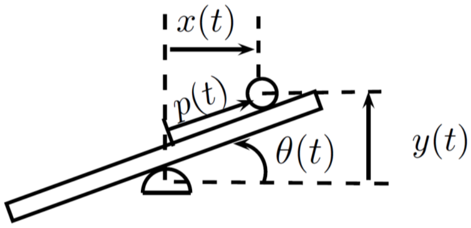

Figure 2: Cartesian Coordinates and Generalized Coordinates

The kinetic energy of the beam is

$$K_1 = \frac{1}{2} J\dot{\theta}^2 \tag{3}$$

where *J* is the moment of inertia of the beam. The kinetic energy of the ball is

$$K_2 = \frac{1}{2} J_b \dot{\theta}_b{}^2 + \frac{1}{2} m v_b{}^2 \tag{4}$$

where $\dot{\theta}_b$ is the angular velocity of the ball and v_b is the linear velocity of the ball. The quantity $\dot{\theta}_b$ can be expressed in terms of the generalized coordinates as

$$\dot{\theta}_b = \frac{p}{r} \tag{5}$$

where *r* is the radius of the ball. We can also express v_b in terms of the generalized coordinates.

$$v_b{}^2 = \dot{x}^2 + \dot{y}^2 \tag{6}$$

$$x = pcos\theta \tag{7}$$

$$\dot{x} = \dot{p}cos\theta - p\dot{\theta}sin\theta \tag{8}$$

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$$\dot{x}^2 = \dot{p}^2cos^2\theta - 2p\dot{p}\dot{\theta}cos\theta sin\theta + p^2\dot{\theta}^2sin^2\theta \tag{9}$$

$$y = psin\theta \tag{10}$$

$$\dot{y} = \dot{p}sin\theta + p\dot{\theta}cos\theta \tag{11}$$

$$\dot{y}^2 = \dot{p}^2sin^2\theta + 2p\dot{p}\dot{\theta}cos\theta sin\theta + p^2\dot{\theta}^2cos^2\theta \tag{12}$$

Substituting (9) and (12) into (6) yields

$$v_b{}^2 = \dot{p}^2 + p^2\dot{\theta}^2 \tag{13}$$

Substituting (5) and (13) into (4), we obtain the expression for the kinetic energy of the ball in terms of the generalized coordinates.

$$K_2 = \frac{1}{2}\left(\frac{J_b}{r^2} + m\right)\dot{p}^2 + \frac{1}{2}mp^2\dot{\theta}^2 \tag{14}$$

The potential energy if the system is given by,

$$U = mgpsin\theta \tag{15}$$

Substituting (3), (14), and (15) into (2), results on the Langrangian for this system

$$L = \frac{1}{2}\left(\frac{J_b}{r^2} + m\right)\dot{p}^2 + \frac{1}{2}(mp^2 + J)\dot{\theta}^2 - mgpsin\theta \tag{16}$$

The first Lagrange equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}}\right) - \frac{\partial L}{\partial p} = 0 \tag{17}$$

We now proceed to compute this equation step-by-step

$$\frac{\partial L}{\partial \dot{p}} = \left(\frac{J_b}{r^2} + m\right)\dot{p} \tag{18}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}}\right) = \left(\frac{J_b}{r^2} + m\right)\ddot{p} \tag{19}$$

$$\frac{\partial L}{\partial p} = mp\dot{\theta}^2 - mgsin\theta \tag{20}$$

Substituting (18)-(20) into (17) we derive the first equation of motion of the ball and beam system

$$\left(\frac{J_b}{r^2}+m\right)\ddot{p}+mgsin\theta-\ mp\dot{\theta}^2\ =0 \quad \textbf{(21)}$$

The second Lagrange equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\ \frac{\partial L}{\partial \theta}=\ \tau \quad \textbf{(22)}$$

Where τ is the external torque applied to the beam. We derive this equation using a similar approach.

$$\frac{\partial L}{\partial \dot{\theta}}=(mp^2+J)\dot{\theta} \quad \textbf{(23)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)=2mp\dot{p}\dot{\theta}+\ (mp^2+J)\ddot{\theta} \quad \textbf{(24)}$$

$$\frac{\partial L}{\partial \theta}=\ -mgpcos\theta \quad \textbf{(25)}$$

Substituting (24) and (25) into (22) we obtain the second equation of motion for the Ball and Beam

$$(mp^2+J)\ddot{\theta}+\ 2mp\dot{p}\dot{\theta}+mgpcos\theta=\ \tau \quad \textbf{(26)}$$

That is, (21) and (26) are the equations of motion for the Ball and Beam system.

3. MODELLING FROM NEWTONIAN MECHANICS

We will now proceed to derive the equations of motion for the Ball and Beam system using Newtonian mechanics. We define the x-axis to be parallel to the beam. Since this axis rotates with time, we must consider the time derivatives of the unit vectors when calculating velocities and accelerations. The absolute acceleration of a body is given by (Merian & Kraige, 2002)

$$a_a=\ \dot{\omega}\ \times\boldsymbol{r}+\ \omega\ \times(\omega\times\boldsymbol{r})+2\omega\ \times\ \boldsymbol{v}_{rel}+\ \boldsymbol{a}_{rel} \quad \textbf{(27)}$$

where ω is the angular velocity of the rotating axis, $\boldsymbol{r}$ is the position vector, $\boldsymbol{v}_{rel}$ is the velocity relative to the rotating axis, and $\boldsymbol{a}_{rel}$ is the acceleration of the body relative to the rotating coordinate system. Consider the set of coordinate axes rotating with an angular velocity of ω shown in Figure 3.

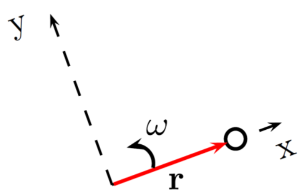

Figure 3: Rotating Axes

Using the same notation used in Figure 1, we can define the necessary vectors

$$\boldsymbol{r} = p\boldsymbol{i} \tag{28}$$

$$\boldsymbol{\omega} = \dot{\theta}\boldsymbol{k} \tag{29}$$

$$\boldsymbol{v}_{rel} = \dot{p}\boldsymbol{i} \tag{30}$$

$$\boldsymbol{a}_{rel} = \ddot{p}\boldsymbol{i} \tag{31}$$

We now proceed to perform the vector products needed to calculate the absolute acceleration (27).

$$\dot{\omega} \times \boldsymbol{r} = \ddot{\theta}\boldsymbol{k} \times p\boldsymbol{i} = p\ddot{\theta}\boldsymbol{j} \tag{32}$$

$$\boldsymbol{\omega} \times \boldsymbol{r} = \dot{\theta}\boldsymbol{k} \times p\boldsymbol{i} = p\dot{\theta}\boldsymbol{j} \tag{33}$$

$$\omega \times (\omega \times \boldsymbol{r}) = \dot{\theta}\boldsymbol{k} \times p\dot{\boldsymbol{\theta}}\boldsymbol{j} = -p\dot{\theta}^2\boldsymbol{i} \tag{34}$$

$$2\omega \times \boldsymbol{v}_{rel} = 2\dot{\theta}\boldsymbol{k} \times \dot{p}\boldsymbol{i} = 2\dot{\theta}\dot{p}\boldsymbol{j} \tag{35}$$

Substituting (31) – (35) into (27) we obtain the acceleration relative to the rotating axes

$$\boldsymbol{a_a} = p\ddot{\theta}\boldsymbol{j} - p\dot{\theta}^2\boldsymbol{i} + 2\dot{\theta}\dot{p}\boldsymbol{j} + \ddot{p}\boldsymbol{i} \tag{36}$$

$$\boldsymbol{a_a} = \left(\ddot{p} - p\dot{\theta}^2\right)\boldsymbol{i} + \left(p\ddot{\theta} + 2\dot{\theta}\dot{p}\right)\boldsymbol{j} \tag{37}$$

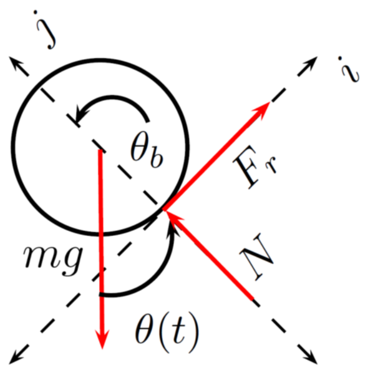

Figure 4: Free Body Diagram of the Rotating Ball

Consider the free body diagram shown in Figure 4. Summing torques about the axis of rotation of the ball yields

$$J_b\ddot{\theta}_b = F_r \cdot r \tag{38}$$

Where J_b is the moment of inertia of the ball about its center. The angle of rotation of the ball about its center θ_b is given by

$$\theta_b = -\frac{p}{r} \tag{39}$$

where r is the radius of the ball. Substituting (39) into (38) and solving for F_r yields

$$F_r = -\frac{J_b}{r^2}\ddot{p}$$

(430)

We now proceed to sum forces acting on the ball in the ***i*** direction. Using the ***i*** component of the relative acceleration vector (37) yields

$$F_r - mgsin\theta = m\left(\ddot{p} - p\dot{\theta}^2\right)$$

(31)

Substituting (40) into (41), we arrive at our first equation of motion

$$\left(\frac{J_b}{r^2} + m\right)\ddot{p} + mgsin\theta - \ mp\dot{\theta}^2 \ = 0$$

(42)

which coincides with (21).

To compute the second equation of motion, we must first compute the normal force N as shown in Figure 4. Summing forces acting on the ball in the **j** direction yields

$$N = m\left(p\ddot{\theta} + 2\dot{\theta}\dot{p}\right) + mgcos\theta$$

(43)

Consider the free body diagram of the beam shown in Figure 5.

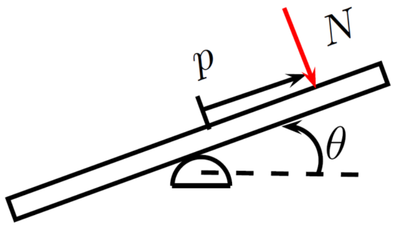

Figure 5: Free Body Diagram of the Beam

Summing torques acting on the beam yields

$$\tau - Np = J\ddot{\theta}$$

(44)

where τ is the external applied torque and J is the inertia of the beam. Substituting (43) into (44) yields our second equation of motion

$$(mp^2 + J)\ddot{\theta} + \ 2mp\dot{p}\dot{\theta} + mgpcos\theta = \ \tau$$

(45)

which coincides with (26) as expected.

The term $mp\dot{\theta}^2$in (42) and the term $2mp\dot{p}\dot{\theta}$ in (45) would be missing if we do not take into consideration the effect of the rotating axis of the beam. These terms are important for nonlinear simulations of the system.

4. NONLINEAR AND LINEAR STATE-VARIABLE REPRESENTATION

The equations of motion we derived for the Ball and Beam system can be written in state-variable representation. We must first define a state vector as follows.

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} p(t) \\ \dot{p}(t) \\ \theta(t) \\ \dot{\theta}(t) \end{bmatrix} \tag{46}$$

This state vector is composed of the minuimum set of variables required to determine the future response of the system given the input and the current state. The equations of motion can then be written in terms of the state variables as

$$\dot{x} = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} x_2 \\ \dfrac{m(x_1 {x_4}^2 - g sin x_3)}{\frac{J_b}{r^2} + m} \\ x_4 \\ \dfrac{-2mx_1x_2x_4 - mgx_1cosx_3 + \tau}{(m{x_1}^2 + J)} \end{bmatrix} = f(x, \tau) \tag{47}$$

We also define an operating point corresponding to a constant ball position p_0 and zero velocity. The corresponding angle and angular velocity of the beam are also zero. The operating state is then given by

$$x_0 = \begin{bmatrix} p_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{48}$$

We also define the operating input required to maintain this operating point, the bancing torque obtained from equating $f(x, \tau)$ in (47) to zero and evaluating at the operating point

$$u_\tau = mgp_0 \tag{49}$$

This is simply the torque required to maintain the ball stationary at position p_0. The Jacobian of the right hand side of (47) with respect to the state vector *x* yields

$$\frac{\partial f}{\partial x}(x, \tau) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{mx_4^2}{\frac{J_b}{r^2} + m} & 0 & \dfrac{-mgcosx_3}{\frac{J_b}{r^2} + m} & \dfrac{2mx_1x_4}{\frac{J_b}{r^2} + m} \\ 0 & 0 & 0 & 1 \\ \dfrac{\partial f_4}{\partial x_1} & \dfrac{-2mx_1x_4}{mx_1^2 + J} & \dfrac{mgx_1sinx_3}{mx_1^2 + J} & \dfrac{-2mx_1x_2}{mx_1^2 + J} \end{bmatrix} \tag{50}$$

where

$$\frac{\partial f_4}{\partial x_1} = \frac{(-2mx_2x_4 - mgcosx_3)(mx_1^2 + J) - (-2mx_1x_2x_4 - mgx_1cosx_3 + \tau)}{(mx_1^2 + J)^2} \tag{51}$$

Evaluatating at the operating point yields the A matrix of the state-variable representation

$$A = \frac{\partial f}{\partial x}(x_0, \tau_0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{-mg}{\dfrac{J_b}{r^2} + m} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{-mg}{mp_0^2 + J} & 0 & 0 & 0 \end{bmatrix}$$

(52)

The Jacobian with respect to the input τ yields

$$\frac{\partial f}{\partial \tau}(x, \tau) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{mx_1^2 + J} \end{bmatrix}$$

(53)

Evaluating at the operating point yields the B matrix of the state variable representation

$$B = \frac{\partial f}{\partial \tau}(x_0, \tau_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{mp_0^2 + J} \end{bmatrix}$$

(54)

Defining the output of the system to be the ball position yields the C matrix of the state variable representation

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

(55)

Using these matrices, the system can now be written in a state-space representation of the form

$$\dot{x}(t) = Ax(t) + Bu(t)$$

(56)

$$y(t) = Cx(t)$$

(57)

Substituting the parameters of the Quanser Consulting Ball and Beam system (Quanser, 2014)

$$m = 0.11kg$$

$$r = 0.015m$$

$$g = 9.81\frac{m}{s^2}$$

$$J = 19 \times 10^{-3}\ kg \cdot m^2$$

$$J_b = 9.99 \times 10^{-6}\ kg \cdot m^2$$

$$p_0 = 0$$

yields the state variable representation

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 1 \\ -56.8 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 52.6 \end{bmatrix} \tau(t)$$

(59)

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(t)$$

(60)

From this state space representation, the system can be analyzed and a controller may be designed.

5. Conclusion

We have derived the dynamical equations for the Ball and Beam system using both Newtonian and Lagrangian mechanics. It has been demonstrated that both methods provide identical results. Some of the terms in these equations are often missed when using Newtonian Mechanics to derive the system equations. This is due to the fact that it is necessary to take into consideration the effect of the rotating axis of the beam. The missing terms arise from the vector products carried out when calculating the absolute acceleration of the ball. These terms introduce additional forces in the equations such as centripetal and Coriolis forces. The energy-based Lagrangian method handles these terms naturally easier. We suggest to us the Lagrangian method in an undergraduate state-space control systems course. Doing so may take from one to two 50 minutes classes to discuss thoroughly. Discussing this method will also provide students with a framework for analyzing other mechanical systems such as the inverted pendulum and the double inverted pendulum. It has been the authors' experience that the Lagrangian method is easier to follow and to understand by students than the Newtonian method.

6. Acknowledgement

This paper is dedicated to the memory of Dr. Andrés Calderón-Colón who inspired and help to develop this work; but could not see the end result due to his early departure last year.

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