Recovering elder´s longitudinal dispersion equation by means of transport coefficient as time function

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ABSTRACT

One of first practical equations for solve with numerical values the advection-dispersion classical model was the Elder´s equation. This relationship was developed regarding the current hydraulic resistance tables for Mountain Rivers but avoiding include the effect on mean velocity due to changes in physical shape of the stream bed, leading to an underestimation of energy losses. A new formula for Longitudinal dispersion coefficient as a time function is presented in this paper allows to apply correctly the old Elder´s formula for conservative tracers which includes the energy slope, and then to know the actual roughness value. An experimental case is examined using ionic tracers.

Keywords: Hydraulics, resistance to flow, tracers.

RESUMEN

Uno de las primeras ecuaciones practicas para resolver con valores numericos el modelo clasico de adveccion-dispersion fue la ecuacion de Elder. Esta relacion fue desarrollada teniendo en cuenta las tablas corrientes de resistencia hidraulica para rios de montaña pero evitando incluir el efecto sobre la velocidad media debido a los cambios en la forma fisica del lecho, llevando a una subestimacion de las perdidas de energia. Una nueva formula para el Coeficiente Longitudinal de dispersion como funcion del tiempo es presentada en este articulo permite aplicar correctamente la formula de Elder para trazadores conservativos que incluye la pendiente de energia, y por lo tanto conocer el valor real de la rugosidad. Un caso experimental es examinado usando trazadores ionicos.

Palabras claves: Hidraulica, resistencia al flujo, trazadores.

1. INTRODUCTION

Only in few past decades specialist have paid attention to a kind of stream roughness component due to some factors as morphological shape of streams bed, playing important role in high slope flows as Mountain Rivers. These complex factors account for high energy losses involved in irreversible process evolving in turbulent flows.

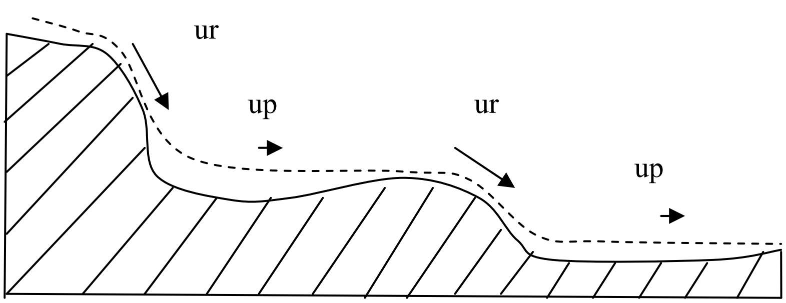


Figure 1: Sequence of rapids and pools in a Mountain stream

Among those factors one may account for turbulence itself, sediment transport, submersion, but mainly for changes in the state of motion of water parcels due to stepped-bed morphology which involve high energy losses. Velocity of those parcels is low in pools and higher in rapids, as in Fig. 1.

These changes between fast and slow flow help to form a sequence of pools and rapids in the stream and reflects irreversible thermodynamic process with high energy losses, as one may see using tracers (Velez Upegui et al, 2004)

2. THE ROLE OF TRACERS CHARACTERIZING THE BEHAVIOR OF FLOW.

Tracers are a special, appropriate marker substance for understand this kind of processes because of two main reasons: A. -Their particles are sensible to several factors affecting flow in water currents reflecting what is occurring in a reach, not in a point as current measurements in hydraulics. B. - The interactions among their molecules may reflect properly what is going on about energy interchange in all fluid in which the tracer is evolving. The second point may be used to examine the energy fate in a pool-rapid scheme, understanding that the tracer behavior may be modeled as a real gas evolving in a denser medium (Van´t Hoff interpretation for tracer expansion). (Constain et al, 2009)

We are talking about the *Joule-Thompson effect* for real gases. In this effect, extensively studied in Nineteen century by several scientist researching van der Waals equation, it is studied the adiabatic expansion of real gases from a recipient *A* to a recipient *B*. Depending on the predominant type or interaction forces, the expanding gas passing from A to B may be heated, increasing its temperature. In this case if predominant interaction forces are of *repulsive* nature, the temperature will increase, proven that there is a thermal isolation in the system (avoiding heat transmission to outside world) Fig. 2.

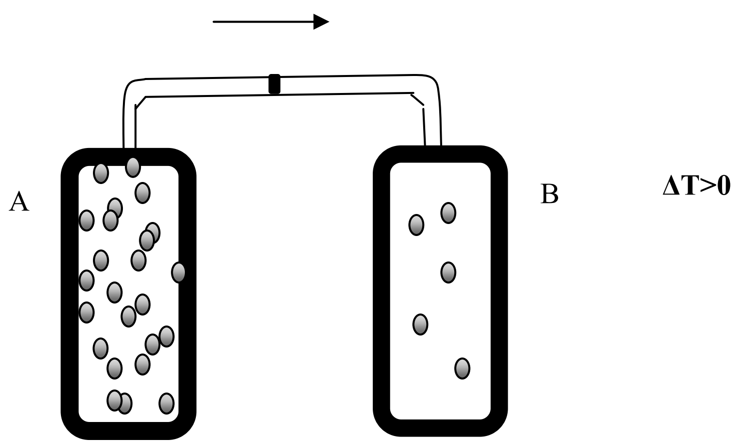


Figure 2: Adiabatic Joule-Thompson Effect in real gasses

Now, changing the nature of the experiments, and allowing a thermal conduction between system and outside world, one may see while the temperature in the system is constant, it should be heat conduction from the system to environments, as in Fig. 3. This means only one thing: That irreversibility has occurred in this expansion process generating internal entropy increment, which has to be ejected as irreversible heat to maintain the isothermal condition. This entropy increase should arise by action of internal friction forces due to interchange of kinetic energy into potential energy and vice versa. This leads to accelerations and decelerations of tracer (fluid) elemental parcels, processes that increase the velocity gradients in the basic viscosity equation (1) where *fF* is the friction force; A is the area in which the force acts tangentially, η the viscosity and u the parcel velocity in X direction.

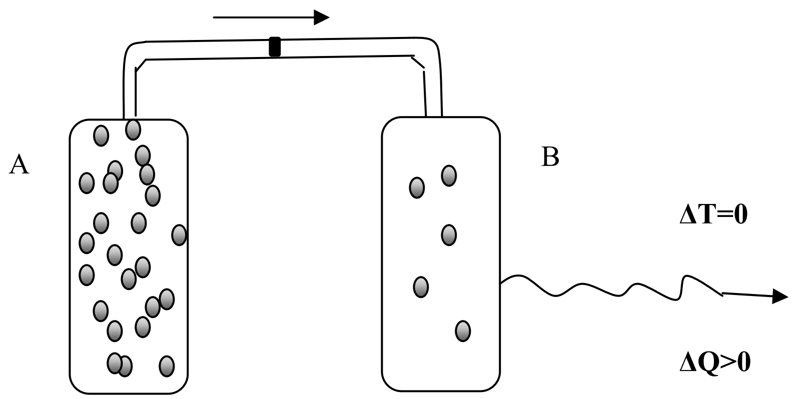


Figure 3: Isothermal Joule-Thompson Effect in real gasses

$$\frac{f_F}{A} = \eta \frac{\partial u}{\partial x} \tag{1}$$

Applying this model to a non-exact but similar scenario composed by a tracer expanding (diffusing) succession of steps in a pool-rapid scheme, the tracer particles will represent the behavior of flow spending energy in irreversible heat form (*Qi*) because of the interchange between kinetic and potential energy and vice versa. Fig. 4

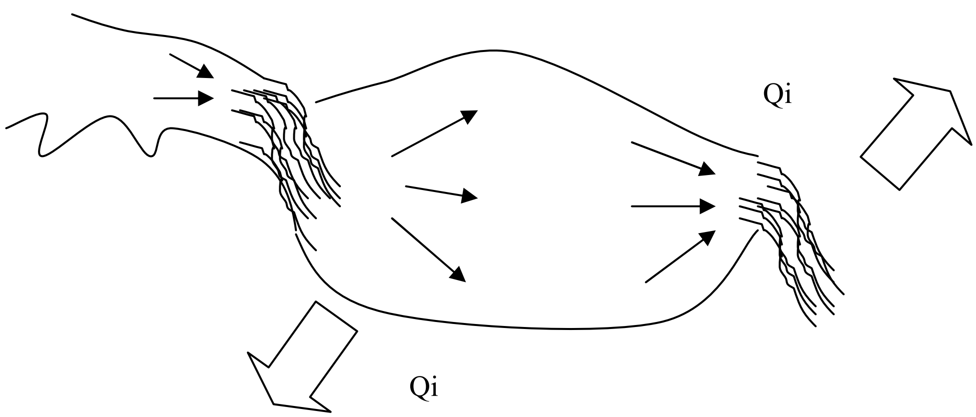


Figure 4: Kinetic and potential energy interchanges producing losses

In the original Joule-Thompson model of heat dissipation of adiabatic real gas expansion, the main driving force for this expansion was the repulsive action related with electronic layers of approaching atoms, while in the tracer cloud the expansion driving force is rather the diffusive effect, several times greater. Whereas the friction forces depend on velocity gradients, and these on velocity distribution in flow, this distribution is a key parameter to describe this energy loss process. Here should be remembered that in turbulence, the dispassion scale for the energy corresponds to very small eddies, at the viscosity level in a Kolmogorov sequence (Peralta-Fabi, 1994).

3. ACTUAL ROUGHNESS VALUE IN MOUNTAIN STREAMS

Regarding the high losses represented by sequential change between kinetic and potential energy in a mountain streams it is convenient to separate different roughness kinds. In equation (2) *n(Grain)* corresponds to current bed friction action and *n(Shape change)* corresponds to the additional losses due to morphology of stream. (Velez Upegui et al, 2004)

$$n_{Total} = n_{Grain} + n_{Shape\ change} \tag{2}$$

It is important to remark that usually the Manning´s number put into Chezy´s equation is viewed as a *point* calculation losing its integral (trench) deep meaning. So it is preferred to put this relationship as function of "trench" variables as discharge, width, mean velocity, and friction slope. All these integral information are well carried by tracers.

$$n_{Total} = \frac{1}{U^{\frac{5}{3}}} \times \left(\frac{Q}{W}\right)^{\frac{2}{3}} \times \sqrt{S} \tag{3}$$

In this enhanced view of roughness it is no difficult to find values of Manning´s number about 0.3 for this kind of streams. As a representative example it is mentioned the several measurements made in a small mountain stream in Medellin, Colombia with a characteristic pool-rapid morphology with values near $n \approx 0.3$ (Velez Upegui et al, 2004).

4. TRACER BEHAVIOR AS A WAY FOR EVALUATE THE FRICTION SLOPE

As was noted before, distribution of velocity in flow is a key factor to evaluate real losses in flow. Additionally, this distribution is useful also to describe shear transport in flows, meaning that both effects have a close relationship. In following paragraphs this role will be examined. G.I. Taylor considering the Reynolds definition for instant velocity as the addition of mean value and velocity deviation, eq.(4) was the first to propose one value of longitudinal transport coefficient function of vertical coordinate (z), depth of flow *d*, and that velocity deviation *u´*, eq.(5) (Fischer, 1966)

$$u = \bar{u} + u' \tag{4}$$

$$E = -d^2 \int_0^1 \left(\frac{z}{d}\right) d\left(\frac{z}{d}\right) \int_0^{z/d'} \frac{1}{\varepsilon_z} d\left(\frac{z}{d}\right) \int_0^{z/d} u' d\left(\frac{z}{d}\right) \tag{5}$$

Using this previous development, J.W. Elder (Fischer, 1966) put a velocity deviation *u´* as function of logarithmic expression and Von Karman constant, κ. (Yuan, 1967)

$$u' = \frac{u^*}{\kappa}\left(1 + Ln\left(\frac{z}{d}\right)\right) \tag{6}$$

Also, with shear velocity u^* as follows, with g gravity acceleration and *R* hydraulic radius:

$$u^* = \sqrt{gRS} \tag{7}$$

Elder finally found the integration of eq. (5) in following terms:

$$E \approx 5.93\ d\ u^* \approx 5.93\ d\ \sqrt{gdS} \tag{8}$$

This expression for the longitudinal shear transport coefficient has great significance up today because of several reasons as: A. It links the general Prandlt velocity profile in turbulent boundary layer with the shear effect of flow as determinant factor in dispersion process, and B. It suggests that Dispersion depending on velocity distribution may be reflecting also the same manifestation of energy dissipation examined in first paragraphs of this paper. If this is true, then eq. (9) that includes energy slope would be congruent with processes of pool-rapid flow type. There was however a set of reasons that forced to abandon the Elder´s equation (presented in 1959) as a valid expression to be widely used in streams. Mainly it was asserted that values of *E* calculated with it were underestimated compared with results of a reference methodology at that time, named "routing procedure" (Fischer, 1966).

5. The Longitudinal dispersion coefficient as time function: A realistic value to recover the Elder´s equation.

The fail of Elder´s equation at that time, despite its deep foundation was an unexpected drawback that leads to search other theoretical ways to define the *E* parameter. Several procedures were proposed along the time considering diversity of effects, some of them with more problems that those supposed to solve.

In this way, there is a conflict when it is proposed the Longitudinal Transport Coefficient *as a constant* in describing shear dispersion in *almost all* current methodologies. This because the only way in which several inertial observers may give a coincident picture of tracer plume evolution (Galilean principle) is using this parameter as function of time. (Constain et al, 2011)

Then if the Longitudinal dispersion Coefficient is time function an Eulerian observer (fixed in border of stream) may compose advective and dispersive velocities to get an asymmetrical curve; at the same time a La Grangian observer do not compose advective velocity because in his coordinate system this velocity does not exist, getting a symmetrical curve. As a matter of fact, using the *same* Fick´s equation one may obtain at the same time *symmetrical* and *asymmetrical* curves putting different conditions.

This consideration leads to suppose that current approaches to the subject, accounting for the transport coefficient *as a constant* invariably, they have limited scope indeed. In this line of thinking, there is no objective reason to guess that is the Elder´s formula the wrong thing and the current methods the correct ones. This result is very important conclusion because applying Elder´s equation it is possible to know the actual value of roughness (Manning´s) resistance to flow of a mountain stream, using the proper *E* value.

To obtain *E(t)* it is necessary to model the relationship between the advective mean velocity, U, and the dispersive velocity, *Vdisp*. This because in such analysis of Galilean composition of velocities this ration always is present.

$$\phi = \frac{V_{disp}}{U} \tag{9}$$

Considering characteristic Gaussian displacement and time parameters (measured in inflection points of bell shaped curve), Δ, τ it holds:

$$V_{disp} = \frac{\Delta}{\tau} \tag{10}$$

If the tracer has a Brownian motion scheme:

$$V_{disp} = \frac{\Delta}{\tau} = \frac{\sqrt{2E\tau}}{\tau} = \sqrt{\frac{2E}{\tau}} \tag{11}$$

Then:

$$U = \frac{1}{\phi}\sqrt{\frac{2E}{\tau}} \tag{12}$$

If dispersion is regarded as a type of "Le Chatelier" reaction mechanism to initial perturbation of the chemical equilibrium, the nature of Φ function should be time dependent because irreversible processes are always put in term of *rate* description (Prigogine et al, 1999). Then, Longitudinal dispersion Coefficient is necessarily a time function:

$$E = \frac{U^2\ \phi(t)^2\,\tau}{2} \tag{13}$$

Then, this accurate definition may be used in Elder´s formula as a right dispersive coefficient.

6. An experimental application of developed formulas in a Mountain river in Colombia.

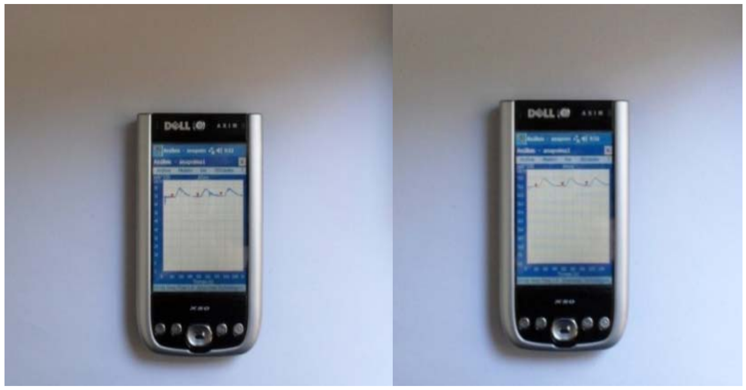
It is described with some details the ionic tracer measurements done in a Colombian mountain river named "Patiño stream" near capital Bogota. This stream show typical pool-rapid scheme for its bed and it is expected that resistance to flow may have a high value, compatible with this type of geomorphology. In Fig.5 photos it may appreciate the several roughness elements affecting the flow, rocks, vegetation, and change of direction, beside the pools and rapids sequence given a high loss of energy factor. Also it is shown the measurement tool.

To do the tracer experiments were used a special hardware-software tool developed by the author´s team, named "Inirida Deep Flow" (IDF), capable to do Rhodamine WT and salt measurements simultaneously. This scientific equipment was used in three parts of the stream, using common salt as tracer in this time.

Figure 5: Some Geomorphologic aspects in "Patiño River" and tracer measurements

6.1 Results of the experiments

Next serial of photos shown screens with tracer curves for the three experiments. In first and second experiment it was injected suddenly three pouring of salt. In last experiment only one pouring was done. The software put red arrows when the tracer is injected. The software also may clean the signal from high frequency noise (as it can be noted in right photos). Fig. 6



First experiment: X= 28 M. M1, 2, 3= 264.8 G (*NaCl*)

Second experiment: X=40 M. M1=264.8 G, M2= 397.2 G, M3=397.2 G (*NaCl*)

Third experiment: X=60 M. M1=251.3 G (*NaCl*)

Figure 6: Screens showing tracer curves with noise and cleaning with filter.

6.2 TABLE OF NUMERICAL VALUES

In next Table 1 it is presented the set of numerical results taken from data stored in IDF software. Discharge value is measured by tracers also and is calculated by the software.

Table1: Data of experiments from IDF instrument

Experiment	**M** **Mass** **(G)**	**τ** **Characteristic dispersion time (S)**	**U** **Mean Velocity (M/s)**	**Q** **Discharge** **(M3/s)**	**Φ**	**Cp** **Peak Concentration (Mgr/l)**	**W** **Width** **(M)**
Trench 1							
Pouring 1 X=28 m	264.8	31.2	0.193	0.511	0.72	3.61	15
Pouring 2 X=28 m	264.8	33.8	0.178	0.692	0.72	3.39	15
Pouring 3 X=28 m	264.8	32.2	0.187	0.538	0.72	3.57	15
Average Trench 1	264.8	32.4	0.186	0.580	0.72	3.52	15
Trench 2							
Pouring 1 X=50 m	264.8	46.1	0.233	0.552	0.60	2.31	10
Pouring 2 X=50 m	397.2	41.2	0.261	0.520	0.60	4.50	10
Pouring 3 X=50 m	397.2	43.7	0.246	0.530	0.60	4.42	10
Average Trench 2		43.7	0.247	0.534	0.60	-------	10
Trench 3							
Pouring 1 X=60 m	251.6	65.1	0.198	0.450	0.58	2.10	10
General	------	---------	0.214	0.542	------	-------	12.1

average	---				---		

Discharge has a statistical dispersion of 8% and also has to be corrected because it has a systematic error due to it is not accomplished yet the "complete mixing" condition. A conservative value is estimated as *Q=0.442 M3/s*, according with some field observations.

6.3. - MASS TRANSPORT CALCULATIONS

For every trench it is calculated the transport coefficient using eq. (13). These data is in Table 2.

$$E(t) = \frac{\phi^2\ U^2 \times \tau}{2}$$

Table 2: Data for calculating *E(t)*.

Average **Measurement**	**τ** **Characteristic dispersion time (S)**	**U** **Mean Velocity** **(M/s)**	**Φ**	**E(t)** **Longitudinal Dispersion Coefficient** **(M2/s)**
Trench 1	32.4	0.186	0.72	0.291
Trench 2	43.7	0.247	0.60	0.480
Trench 3	65.1	0.198	0.58	0.502
General average	-------	0.214	-------	0.424

6.4 FLOW RESISTANCE (ROUGHNESS) CALCULATIONS

Mean hydraulic area of cross section is:

$$A = \frac{Q}{U} \approx \frac{0.542}{0.214} \approx 2.53m^2$$

Then, mean depth in trench is:

$$d \approx \frac{A}{W} = \frac{2.53}{12.1} \approx 0.21m$$

The mean hydraulic radio is:

$$R = \frac{A}{(2d + W)} = \frac{2.53}{(2 \times 0.21 + 12.1)} \approx 0.20m$$

Now, assuming a predominant shear effect due to vertical velocity distribution, applying Elder´s formula with the mean *E(t)* value of Table 2, it holds:

$$E \approx 5.93\ d\ U^* = 5.93\ d\ \sqrt{gdS}$$

Clearing S, the friction slope:

$$S \approx \frac{1}{35.2} \times \frac{E^2}{g\ d^3} = \frac{1}{35.2} \times \frac{0.424^2}{9.81 \times (0.21)^3} \approx 0.056$$

This slope corresponds to height of 5.6 m in 100 m.

Now, the Chezy´s coefficient is:

$$C = \frac{U}{\sqrt{RS}} \approx \frac{0.214}{\sqrt{0.20 \times 0.056}} \approx 2.02\ m^{\frac{1}{2}} / s$$

Using Manning´s formula approximately:

$$n \approx \frac{R^{\frac{1}{6}}}{C} = \frac{(0.20)^{1/6}}{2.02} \approx 0.379$$

This high value of Manning´s number is compatible with the high roughness of the river but with the presence of pool-rapid schemes which lead to large energy losses in flow, in the same level those other mountain rivers in Colombia.

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